Integrálás

Határozatlan (Anti-derivált)

$f(x)$ fgv prim. fgv-e $F(x)$, ha $F'(x) = f(x)$ $\forall x \in$ vmi intervallum.
ha F prim fgv $\Rightarrow$ $F+C$ prim. fgv $(c \in \mathbb{R})$, ebben az esetben
$\int f = \{F + c;\ c \in \mathbb{R}\}$

a, c konstans, $f, g: \mathbb{R} \to \mathbb{R}$ fgv

- $\int a f(x) + c g(x)\,dx = a \cdot \int f(x)\,dx + c \cdot \int g(x)\,dx$
- $\mu \in \mathbb{R} - \{-1\}$ $\int x^{\mu}\,dx = \frac{x^{\mu+1}}{\mu+1} + c$
- $\int x^{-1}\,dx = \ln|x| + c$
- $\int \sin(x)\,dx = -\cos(x) + c$, $\int \cos(x)\,dx = \sin(x) + c$,
  $\int e^x\,dx = e^x + c$, $\int N^x\,dx = \frac{1}{\ln(N)} N^x + c$

$$\int f(x)^{\mu} \cdot f'(x)\,dx = \frac{f^{\mu+1}(x)}{\mu+1} + c$$
$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| - c$$

1. FELADAT

a) $\int 6x^7 + 9x^4\,dx = 6\frac{x^8}{8} + 9\frac{x^5}{5} + c$

b) $\int \frac{3x^4 - 2x^2}{\sqrt[3]{x}}\,dx = 3\frac{x^{4+2/3}}{4+2/3} - 2\frac{x^{2+2/3}}{2+2/3} + c$

c) $\int 7^{x-3}\,dx = \frac{7^{-3}}{\ln(7)} \cdot 7^x + c$

d) $\int 7\sin(x) - e^x\,dx = -7\cos(x) - e^x + c$

2. FELADAT

Intro: $\int \frac{12x^2 - 4x + 6}{4x^3 - 2x^2 + 6x - 8}\,dx = \ln|4x^3 - 2x^2 + 6x - 8| + c$

a) $\int \frac{9x^2}{3x^3 - 7}\,dx = \ln|3x^3 - 7| + c$

b) $\int \frac{x^2}{3x^3 - 7}\,dx = \frac{1}{9} \cdot \int \frac{9x^2}{3x^3 - 7}\,dx = \frac{1}{9} \cdot \ln|3x^3 - 7| + c$

d) $\int \frac{1}{x \cdot \ln(x)}\,dx = \int \frac{\frac{1}{x}}{\ln(x)}\,dx = \ln|\ln(x)| + c$

e) $\int (x+7)^{12}\,dx = \frac{(x+7)^{13}}{13} + c$

f) $\int \sin^9(x) \cdot \cos(x)\,dx = \frac{\sin^{10}(x)}{10} + c$

g) $\int x\sqrt{3 + 2x^2}\,dx = \frac{(3 + 2x^2)^{3/2}}{3/2} + c$

$$\int f(ax+b)\,dx = \frac{1}{a} F(ax+b) + c$$

3. FELADAT

a) $\int \sin(2x + \pi/2)\,dx = -\cos(2x + \pi/2) \cdot \frac{1}{2} + c$

b) $\int 4^{3x+7}\,dx = \frac{1}{\ln(4)} \cdot 4^{3x+7} + c$

$\left(\operatorname{arctg}'(x) = \frac{1}{1+x^2} \quad \arcsin'(x) = \frac{1}{\sqrt{1-x^2}} \right)$

c) $\int \frac{1}{4x^2+28x+50}\,dx = \int \frac{1}{1+(2x+7)^2}\,dx = \frac{\operatorname{arctg}(2x+7)}{2}$

d) $\int \frac{1}{3+4/3\,x^2}\,dx =$

$$\boxed{\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)\,v(x)\,dx}$$

$\boxed{\text{4. FELADAT}}$

a) $\int \underset{u}{x} \cdot \underset{v'}{\cos(x)}\,dx = x \cdot \sin(x) - \int \sin(x)\,dx = x \cdot \sin(x) + \cos(x) + c$

$u' = 1 \quad v = \sin(x)$

b) $\int \underset{u}{(3x^2-2x+6)}\underset{v'}{e^x}\,dx = (3x^2-2x+6)e^x - \int (6x-2)e^x dx = (3x^2-2x+6)e^x$

$u' = 6x-2 \quad v = e^x$

$- \int \underset{u}{(6x-2)}\underset{v'}{e^x} dx = (3x^2-2x+6)e^x - \left[\underset{u \cdot v}{(6x-2)e^x} - \underset{\int u' \cdot v}{6 \cdot e^x}\right] + c$

$u' = 6 \quad v = e^x$

*c) $\int \underset{u'}{x^k} \underset{v}{\ln(x)}\,dx = \frac{x^{k+1}}{k+1}\ln(x) - \int \frac{x^k}{k+1}\,dx = \frac{x^{k+1}}{k+1}\ln(x) - \int \frac{x^k}{k+1}\,dx =$

$u = x^{k+1}/k+1 \quad v' = 1/x$

$= \frac{x^{k+1}}{k+1}\ln(x) - \frac{x^{k+1}}{(k+1)^2} + c$

d) $\int \underset{u}{\sin(x)} \cdot \underset{v'}{\cos(x)}\,dx = \sin^2(x) - \int \cos(x)\sin(x)\,dx$

$u' = \cos(x) \quad v = \sin(x)$

$\Rightarrow \int \cos(x)\sin(x)\,dx = \frac{\sin^2(x)}{2} + c = \frac{1-\cos^2(x)}{2} + c = -\frac{\cos^2(x)}{2} + c$

e) $\int \underset{u'}{e^x} \underset{v}{\cos(x)}\,dx = e^x\cos(x) + \int \underset{u'}{e^x} \underset{v}{\sin(x)}\,dx = e^x(\cos(x)+\sin(x)) - \int e^x\cos(x)\,dx$

$u = e^x \quad v' = -\sin(x) \qquad u = e^x \quad v' = \cos(x)$

$\Rightarrow \int e^x\cos(x)\,dx = \frac{e^x(\cos(x)+\sin(x))}{2} + c$

f) $\int \operatorname{arctg}(x)\,dx = \int \underset{u'}{1} \cdot \underset{v}{\operatorname{arctg}(x)}\,dx = x \cdot \operatorname{arctg}(x) - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx = x\operatorname{arctg}(x) - \frac{1}{2}\ln(1+x^2) + c$

$u = x \quad v' = \frac{1}{1+x^2}$

g) $\int \arcsin(x)\,dx = \int 1 \cdot \arcsin(x)\,dx = x\arcsin(x) + \frac{1}{2}\int (-2x)(1-x^2)^{-1/2}\,dx$

$= x \cdot \arcsin(x) + (1-x^2)^{1/2} + c$

h) $\int \cos^n(x) = \int \underset{u}{\cos^{n-1}x}\ \underset{v'}{\cos x}\,dx = \cos^{n-1}(x)\cdot\sin(x) + (n-1)\int \cos^{n-2}(x)\underbrace{\sin^2(x)}_{1-\cos^2(x)}dx =$

$u' = -(n-1)\cos^{n-2}x \sin(x) \quad v = \sin x$

$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx$

$-(n-1)\int \cos^n(x)\,dx$

$\Rightarrow \int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{(n-1)}{n}\int \cos^{n-2}(x)\,dx$

i) $\int \sin^n(x)$

$$\sin^2 x + \cos^2 x = 1$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \cdot \tan y}$$
$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$
$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$
$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$
$$\sin x \sin y = -\frac{1}{2}[\cos(x+y) - \cos(x-y)]$$

5. FELADAT trigon. fgv-ek linearizálással

a) $\int \sin^2(x)dx = \int \frac{1-\cos(2x)}{2}dx = \frac{1}{2}x - \int \frac{\cos(2x)}{2} = \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right) + c$

b) $\int \cos^3(x)dx = \int (1-\sin^2(x))\cos(x)dx = \sin(x) - \frac{\sin^3(x)}{3} + c$

c) $\int \sin(x)\cdot\sin(2x)dx = 2\int \underbrace{\sin^2(x)}_{f^2(x)}\underbrace{\cos(x)}_{f'(x)}dx = 2\,\frac{\sin^3(x)}{3} + c$

d) $\int \sin^3(x)\cos^2(x)dx = \int \sin(x)(1-\cos^2(x))\cos^2(x)dx =$
$= -\frac{\cos^3(x)}{3} + \frac{\cos^5(x)}{5} + c$

e) $\int \sin^5(x)\cos^2(x)dx = \int \sin(x)\cos^2(x)\underbrace{(1-\cos^2(x))^2}_{1-2\cos^2(x)+\cos^4(x)}dx = \frac{-\cos^3(x)}{3} + \frac{2}{5}\cos^5(x) - \frac{\cos^7(x)}{7} + c$

VEGYES FELADATOK

a) $\int \sin^3(7x)\cos(7x)dx$

b) $\int \frac{\sin^3(x)}{1-\cos(x)}dx$

c) $\int (2x-3)^{100}dx$

d) $\int \sin^3(x)dx$

e) $\int \frac{\ln^2(x)}{x}dx$

f) $\int \frac{1}{\sqrt{\sin(x)\cos^3(x)}}dx$

g) $\int \frac{1}{x^3\sqrt{\ln(x)}}dx$

h) $\int \frac{x}{\sqrt{1-x^2}}dx$

i) $\int \sqrt[3]{1-3x}\,dx$

j) $\int \frac{e^{2x}}{1+e^{2x}}dx$

k) $\int \frac{x}{x^2+a^2}dx$

l) $\int \frac{1}{\mathrm{sh}(x)}dx$

m) $\int \frac{dx}{2+3x^2}$

n) $\int \frac{1}{3x^2+6x+1}dx$

o) $\int e^x \sin^2(x)dx$

a) $\int \underbrace{\sin^3(7x)}_{f^3(7x)} \underbrace{\cos(7x)}_{\cdot f'(7x)} dx = \frac{\sin^4(7x)}{7 \cdot 4} + c$

b) $\int \frac{\sin^3(x)}{1-\cos(x)} dx = \int \frac{\sin(x)\cancel{(1-\cos(x))}(1+\cos(x))}{\cancel{1-\cos(x)}} dx = -\cos(x) + \int \underbrace{\sin(x)}_{f^1} \cdot \underbrace{\cos(x)}_{f'} d =$

$= -\cos(x) + \frac{\sin^2(x)}{2} + c$

c) $\int (2x-3)^{100} dx = \frac{(2x-3)^{101}}{101} + c$

d) $\int \sin^3(x)\, dx = \int \sin(x) \cdot (1-\cos^2(x))\, dx = -\cos(x) + \frac{\cos^3(x)}{3} + c$

e) $\int \frac{\ln^2(x)}{x} dx = \int \ln^2(x) \cdot \frac{1}{x} dx = \frac{\ln^3(x)}{3} + c$

f) $\int \frac{1}{\sqrt{\sin(x)\cos^3(x)}} dx =$

g) $\int \frac{1}{x\sqrt[3]{\ln(x)}} dx =$

h) $\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int -2x(1-x^2)^{-1/2} dx = -(1-x^2)^{1/2}$

i) $\int \sqrt[3]{1-3x}\, dx = -\frac{1}{3}\int (-3 \cdot 1)(1-3x)^{1/3} dx = (1-3x)^{4/3} \cdot \frac{3}{4} + c$

j) $\int \frac{e^{2x}}{1+e^{2x}} dx = \frac{\ln|1+e^{2x}|}{2} + c$

k) $\int \frac{2x}{x^2+a^2} dx = \ln|x^2+a^2| + c$

l) $\int \frac{1}{sh(x)} dx =$

m) $\int \frac{dx}{2+3x^2} = \frac{1}{2}\int \frac{1}{1+(\frac{\sqrt{3}}{\sqrt{2}}x)^2} dx = \frac{1}{\sqrt{6}} \operatorname{arctg}(\frac{\sqrt{3}}{\sqrt{2}}x)$

n) $\int \frac{1}{3x^2+6x+3} dx = \int \frac{1}{3(x^2+2x+1)} dx = \frac{1}{3} \cdot \int \frac{1}{(x+1)^2} dx = -\frac{1}{3}(x+1)^{-1}$

o) $\int e^x \sin^2(x)\, dx$