Határozott integrál (Riemann integrál)

Ha f folyt $[a,b]$-n $\Rightarrow$ Riemann integrálható $[a,b]$-n

és

$$\int_a^b f(x)\,dx = F(b)-F(a) = [F(x)]_a^b \qquad (F'(x)=f(x) \quad x\in[a,b])$$

Pl. $\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

ALKALMAZÁSOK:

ÍVHOSSZ:

Ha f $[a,b]$-n folytonosan diffható $\rightarrow$ $y=f(x)$ ívhossza

$$\int_a^b \sqrt{1+f'^2(x)}\,dx = s$$

Pl. $f(x) = 1-\frac{x^2}{4}$, $x\in[0,2]$

$f'(x) = -\frac{x}{2}$ $\Rightarrow$ $s = \int_0^2 \sqrt{1+\frac{x^2}{4}}\,dx = \int_0^{\operatorname{arsh}(1)} \sqrt{1+\operatorname{sh}^2(t)}\cdot 2\operatorname{ch}(t)\,dt =$

$$= 2\int_0^{\operatorname{arsh}(1)} \operatorname{ch}^2(t)\,dt = \left[\frac{\operatorname{sh}(2t)}{2} + t\right]_0^{\operatorname{arsh}(1)} = \operatorname{sh}(\operatorname{arsh}(1))\cdot\operatorname{ch}(\operatorname{arsh}(1)) + \operatorname{arsh}(1) - \operatorname{sh}(0)\operatorname{ch}(0) - 0 = \boxed{\sqrt{2}+\operatorname{arsh}(1)}$$

$2\operatorname{sh}(t)\operatorname{ch}(t) = 2\operatorname{sh}(\operatorname{arsh}(1))\cdot\operatorname{ch}(\operatorname{arsh}(1))$

$\operatorname{ch}(\operatorname{arsh}(x)) = \sqrt{1+x^2}$

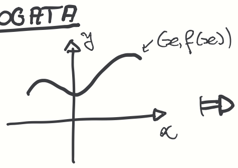
$\frac{x}{2} = \operatorname{sh}(t) \Rightarrow t = \operatorname{arsh}(\frac{x}{2})$ $\quad x=0 \rightarrow t=0$; $\quad x=2 \Rightarrow \operatorname{arsh}(1)=t$

$x = 2\operatorname{sh}(t)$

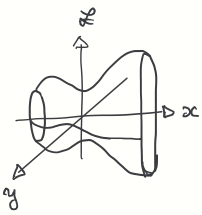
$\frac{dx}{dt} = 2\operatorname{ch}(t)$

FORGÁSTEST TÉRFOGATA

Adott $f(x)$ megforgatjuk



$\Rightarrow$



$V = \pi\cdot\int f^2(x)\,dx$

Pl. $f(x) = \sqrt{1+x^2}$ $\quad x\in[0,3]$

$$V = \pi\cdot\int_0^3 1+x^2\,dx = \pi\left[x+\frac{x^3}{3}\right]_0^3 = \boxed{12\pi}$$

FELSZÍN

A forgástest felszíne: $\quad s = 2\pi\cdot\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx \qquad r = f(x)\cdot\sqrt{1+f'^2(x)}$

Pl. $\operatorname{tg}(x)$, $x\in[0,\frac{\pi}{4}]$.

$$s = 2\pi\cdot\int_0^{\pi/4} \frac{\sin(x)}{\cos(x)}\cdot\sqrt{1+\frac{1}{\cos^4(x)}}\,dx = -\pi\cdot\int_0^{\pi/4}\frac{-2\sin(x)}{\cos(x)}\sqrt{1+\frac{1}{\cos^4(x)}}\,dx = -2\pi\cdot\int_{\operatorname{arsh}(1)}^{\operatorname{arsh}(2)}\operatorname{ch}(u)\sqrt{1+\operatorname{sh}^2(u)}\,du = -2\pi\int_{\operatorname{arsh}(1)}^{\operatorname{arsh}(2)}\operatorname{ch}^2(u)\,du =$$

$\cos^{-2}(x) = \operatorname{sh}(u)$

$-2\cos^{-1}(x)\sin(x)\,dx = \operatorname{ch}(u)\,du$

$\int f(g(x))\cdot g'(x)\,dx = \int f(h(u))\cdot h'(u)\,du$

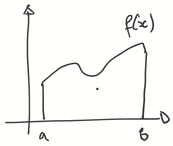
$$= -2\pi\int_{\operatorname{arsh}(1)}^{\operatorname{arsh}(2)}\frac{\operatorname{ch}(2u)+1}{2}\,du = -2\pi\left[\frac{\operatorname{sh}(2u)}{4}+\frac{u}{2}\right]_{\operatorname{arsh}(1)}^{\operatorname{arsh}(2)}$$

$\operatorname{ch}^2(u) = \frac{\operatorname{ch}(2u)+1}{2}$

$$= \boxed{-\pi\left(2\sqrt{5}+\operatorname{arsh}(2) - 2\cdot 1\cdot\sqrt{2} - \operatorname{arsh}(1)\right)}$$

$\operatorname{sh}(2\operatorname{arsh}(y)) = 2\operatorname{sh}(\operatorname{arsh}(y))\cdot\operatorname{ch}(\operatorname{arsh}(y)) = 2y\sqrt{1+y^2}$

TÖMEGKP



(x_s, y_s), ahol

$$x_s = \frac{M_y(a,b)}{T(a,b)} \qquad y_s = \frac{M_x(a,b)}{T(a,b)} \qquad M_y(a,b) = \int_a^b x \cdot f(x)\,dx$$

$$T(a,b) = \int_a^b f(x)\,dx \qquad M_x(a,b) = \frac{1}{2}\int f(x)^2\,dx$$

Pl. $f(x) = x^3$, $a = 0$, $b = 1$

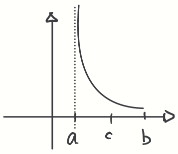
- $T(a,b) = \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$
- $M_x(a,b) = \frac{1}{2}\int_0^1 x^6\,dx = \left[\frac{x^7}{7}\right]_0^1 = \frac{1}{14}$ $\Rightarrow$ $\boxed{x_s = \frac{4}{5} \quad y_s = \frac{4}{14}}$
- $M_y(a,b) = \int_0^1 x^4\,dx = \left[\frac{x^5}{5}\right] = \frac{1}{5}$

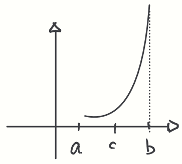
Improprius integrál

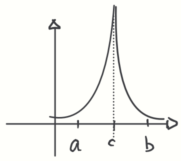
Eml.: Ha f korlátos fgv folytonos az $[a,b]$ korlátos intervallumon kivéve max. megszámlálható sok pontban $\Rightarrow$ f Riemann integrálható $[a,b]$-n.

Hol romlik ez el?

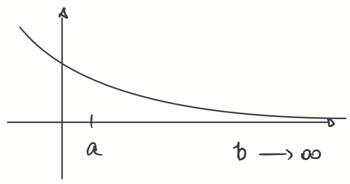
A/ Nem korlátos a fgv:







B/ Nem korlátos int.



Df. IMPROPRIUS INTEGRÁL

Ha f folyt $(a,b]$-n és nem korl. ha $x \to a+0$
$$\int_a^b f(x)\,dx = \lim_{d \to a+0} \int_d^b f(x)\,dx$$

Ha f folyt $(a,b]$-n és nem korl. ha $x \to a+0$
$$\int_a^b f(x)\,dx = \lim_{d \to b-0} \int_a^d f(x)\,dx$$

Ha f folyt $[a,c)$-n és $(c,b]$-n nem korl., ha $x \to c$
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{d \to c-} \int_a^d f(x)\,dx + \lim_{d \to c+} \int_d^b f(x)\,dx$$

Ha f folyt $[a,\infty)$-en
$$\int_a^\infty f(x)\,dx := \lim_{d \to \infty} \int_a^d f(x)\,dx$$
Ha f folyt $(-\infty,b]$-n
$$\int_{-\infty}^b f(x)\,dx = \lim_{d \to -\infty} \int_d^b f(x)\,dx$$
Ha f folyt $(-\infty,\infty)$-en
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{c \to \infty \\ d \to -\infty}} \int_{-\infty}^{\infty} f(x)\,dx$$

Ha ezek a határértékek konvergálnak akkor az improprius integrál konvergens, ha nem, akkor divergens.

Pl.

1. $$\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{d \to 2-} \int_0^d \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{d \to 2-} \frac{1}{2} \int_0^d \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx = \lim_{d \to 2-} \left[\arcsin\left(\frac{x}{2}\right)\right]_0^d = \lim_{d \to 2-} \arcsin\left(\frac{d}{2}\right) - \arcsin(0) = \boxed{\frac{\pi}{2}}$$ ☺

   ↳ folytonos $[0,2)$-n nem korl. ha $x \to 2-$

   Konvergens!

2. $$\int_{-3}^0 \frac{1}{\sqrt[5]{(x+2)^4}}\,dx = \lim_{d \to -2-} \int_{-3}^d \frac{1}{\sqrt[5]{(x+2)^4}} + \lim_{d \to -2+} \int_d^0 \frac{1}{\sqrt[5]{(x+2)^4}}\,dx = 5 \cdot \left[\lim_{d \to -2-} \left[(x+2)^{\frac{1}{5}}\right]_{-3}^d + \lim_{d \to -2+} \left[(x+2)^{\frac{1}{5}}\right]_d^0\right] = \boxed{5 \cdot \left(1 + \sqrt[5]{2}\right)}$$

   ↳ nem korl. az $x=-2$-ben

   $$\int (x+2)^{-\frac{4}{5}}\,dx = (x+2)^{\frac{1}{5}} \cdot 5$$

   Konvergens

   Konvergens!

3. $$\int_0^2 \frac{1}{(x-1)}\,dx = \lim_{d \to 1-} \int_0^d \frac{1}{x-1}\,dx + \lim_{d \to 1+} \int_d^2 \frac{1}{x-1}\,dx = \lim_{d \to 1-} \left[\ln|x-1|\right]_0^d + \lim_{d \to 1+} \left[\ln|x-1|\right]_d^2$$

   „$(-\infty)-0$" „$0-(-\infty)$"

   Divergens!

4. $$\int_0^\infty e^{-x}\,dx = \lim_{d \to \infty} \int_0^d e^{-x}\,dx = \lim_{d \to \infty} \left[-e^{-x}\right]_0^d = \lim_{d \to \infty} \left[-e^{-d} + e^{-0}\right] = +1$$

   Konvergens!

5. $\displaystyle\int_1^\infty x^{-p}\,dx = \lim_{d\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^{\infty} \underset{\Uparrow}{=} \begin{cases}\frac{1}{p-1}, & \text{ha } p>1\\ \text{div.}, & \text{ha } p\le 1\end{cases}$

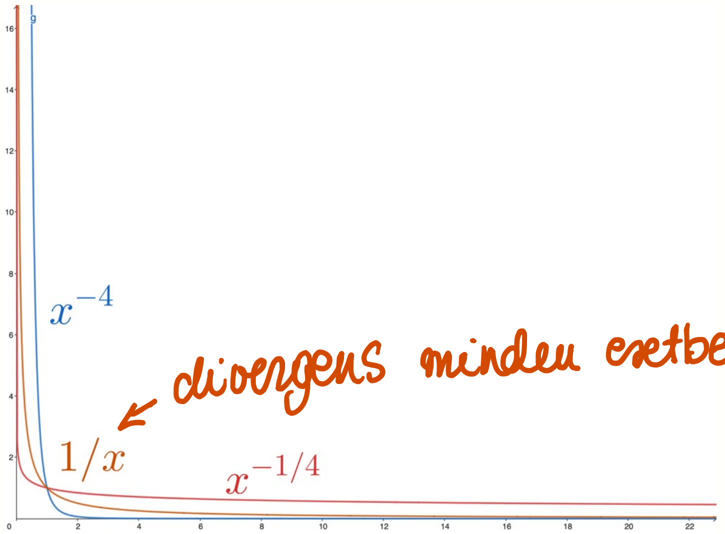
$p\neq 1$

$\lim_{d\to\infty} x^{e}$ pontosan akkor konvergens (h.é. $=0$), ha $e<0$, tehát, a fenti esetben, ha $-p+1<0 \Leftrightarrow p>1$

6. $\displaystyle\int_0^1 x^{-p}\,dx = \lim_{d\to 0}\left[\frac{x^{-p+1}}{-p+1}\right]_d^{1} = \begin{cases}\frac{1}{1-p}, & \text{ha } p<1\\ \text{divergens egyébként}\end{cases}$

$p\neq 1$

$\lim_{d\to 0} x^{d}$ konv $\Leftrightarrow d\geq 0$, a fenti esetben, ha $-p+1\geq 0 \Leftrightarrow p<1$



divergens minden esetben

$\boxed{p=1}$:

$$\int_0^1 \frac{1}{x}\,dx = \lim_{d\to 0+}\big[\ln|x|\big]_d^1 = 0 - \underbrace{\lim_{d\to 0+}\ln|d|}_{-\infty}$$

divergens.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{d\to\infty}\big[\ln|x|\big]_1^d = \lim_{d\to\infty}\ln|d| - 0 =$$

$$=\infty$$

divergens.