MASTER THESIS

The projection of the random Menger sponge

Vilma Orgoványi

Supervisor: Prof. Károly Simon

Professor of Mathematics

BME Institute of Mathematics

Department of Stochatics

BME

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Contents

Chapter 1

Introduction

Considerable attention has been payed to the case when all the probabilities in the construction of the Mandelbrot percolation (see the Mandelbrot percolation section) are identical. However, in this thesis we focus on the much more difficult case when the probabilities are not identical. We concentrate on the orthogonal projection of the three-dimensional Mandelbrot percolation to the line $\{t\underline{e} \mid t \in \mathbb{R}\}$, where $\underline{e} = (1,1,1)$. The methods we use prove statements about the projection are very similar to the ones in [4] and [6], although in these articles the authors consider the sum of two independent one-dimensional Mandelbrot percolations. The problem in our case is easier since we have less dependence in our construction. The main difficulty was to find the way how we could use similar arguments in the three- instead of two-dimensional case. Our main example throughout this thesis is the random Menger sponge (defined in the section Menger sponge). The structure of the thesis is as follows.

Later in this chapter (chapter 1) we define the Mandelbrot percolation

fractal, the Menger sponge and the random Menger sponge.

In the second chapter we define conditions, and prove that under these conditions the projection of the Mandelbrot percolation contains an interval almost surely, following the lines of [4]. Later we prove that in the case of the random Menger sponge the conditions are also necessary.

In chapter 3 we define conditions under which the projection of the Mandelbrot percolation has positive Lebesgue measure almost surely analogously to [6].

In the last – Conclusion chapter we summarize the results and show an interesting consequence regarding the attractor of a random self-similar iterated function system on the line.

1.1 Mandelbrot percolation

1.1.1 The intuition

For the d-dimensional Mandelbrot percolation we choose an integer M. Then we take the d-dimensional unit square, and divide it to M^d congruent subsquares, for each subsquares we choose a probability, which remains the same until the end of the process and keep each square with the assigned probability. For the retained squares we repeat the same process, and so on, if we do this infinitely many times we get the so-called Mandelbrot percolation fractal.
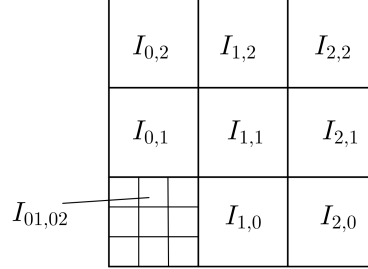
Figure 1.1: The partition in the case of the two-dimensional Mandelbrot percolation.

1.1.2 Construction of the Mandelbrot percolation fractal

The following construction is from [3]. Let $I := [0,1]^d$ denote the unit square. For given $M \geq 2$ integer and $p_{i_1,\dots,i_d} \in [0,1]$ for $(i_1,\dots,i_d) \in \{0,1,\dots,M-1\}^d$ probabilities the Mandelbrot percolation set in the d-dimensional Euclidean-space is constructed in the following way: Let $\mathcal{T}_n := \{(\underline{i}_{1_n},\dots,\underline{i}_{d_n}) \mid \underline{i}_{1_n},\dots,\underline{i}_{d_n} \in \{0,1,\dots,M-1\}^n\}$ denote the d-lets of sequences of length n from $\{0,1,\dots,M-1\}$ indexing the level n sub-squares of I, the empty sequence is denoted by $\emptyset$, as follows $\mathcal{T}_0 = (\emptyset,\dots,\emptyset)$. Denote the first level sub-squares of I of side length $\dfrac{1}{M}$ with $I_{i_1,\dots,i_d}$:

$$I_{i_1,\dots,i_d} := \left[\frac{i_1}{M},\frac{i_1+1}{M}\right] \times \left[\frac{i_2}{M},\frac{i_2+1}{M}\right] \times \cdots \times \left[\frac{i_d}{M},\frac{i_d+1}{M}\right] \tag{1.1}$$

This is a partition of the unit square into M^d congruent squares :

$$I = \bigcup_{i_1,\ldots,i_d=0}^{M-1} I_{i_1,\ldots,i_d}.$$

We can define the level n squares similarly: if $(\underline{i}_{1_n}, \ldots, \underline{i}_{d_n}) \in \mathcal{T}_n$ then

$$I_{\underline{i}_{1_n},\ldots,\underline{i}_{d_n}} = \left[\sum_{k=1}^{n} i_{1_k} \cdot \frac{1}{M^k}, \sum_{k=1}^{n} i_{1_k} \cdot \frac{1}{M^k} + \frac{1}{M^n} \right] \times \ldots$$

$$\times \left[\sum_{k=1}^{n} i_{d_k} \cdot \frac{1}{M^k}, \sum_{k=1}^{n} i_{d_k} \cdot \frac{1}{M^k} + \frac{1}{M^n} \right]. \quad (1.2)$$

Now we have the base for the fractal percolation set. The next step is to define the survival set $\mathcal{E}_n$ consisting the indices of the retained level n squares.

Definition 1.1. $\mathcal{E}_0 = \mathcal{T}_0 = (\emptyset, \ldots, \emptyset)$ *and inductively if we have* $\mathcal{E}_{n-1}$ *and* $(\underline{i}_{1_{n-1}}, \ldots, \underline{i}_{d_{n-1}}) \notin \mathcal{E}_{n-1}$ *then for all* $(j_1, \ldots, j_d) \in \{0, 1, \ldots, M - 1\}^d$ $(i_{1_1} \ldots i_{1_{n-1}} j_1, \ldots, i_{d_1}, \ldots, i_{d_{n-1}}, j_d) \notin \mathcal{E}_n$, *if* $(\underline{i}_{1_{n-1}}, \ldots, \underline{i}_{d_{n-1}}) \in \mathcal{E}_{n-1}$ *then* $(i_{1_1} \ldots i_{1_{n-1}} j_1, \ldots, i_{d_1} \ldots i_{d_{n-1}} j_d) \in \mathcal{E}_n$ *with probability* $p_{j_1,\ldots,j_d}$.

We can also think about $\mathcal{T}_n$ as an M^d-ary tree with height n and nodes $(\underline{i}_{1_k}, \ldots, \underline{i}_{d_k})$. An $(\underline{i}_{1_k}, \ldots, \underline{i}_{d_k})$ node has M^d children: $(\underline{i}_{1_k} j_1, \ldots, \underline{i}_{d_k} j_d)$, $j_1, \ldots, j_d \in \{0, \ldots, M - 1\}$. For $\underline{p} = (p_{0,\ldots,0}, \ldots, p_{M-1,\ldots,M-1})$ we can introduce a probability measure $\mathbb{P}_{\underline{p}}$ on the space of labeled trees. For each node $(i_{1_1} \ldots i_{1_n}, \ldots, i_{d_1} \ldots i_{d_n})$ we give a random label $X_{i_{1_1}\ldots i_{1_n},\ldots,i_{d_1}\ldots i_{d_n}}$ this will be 0 or 1. It is required that

1. $X_{i_{1_1}\ldots i_{1_n},\ldots,i_{d_1}\ldots i_{d_n}}$ are independent Bernoulli random variables;

2. $\mathbb{P}(X_\emptyset) = 1$;

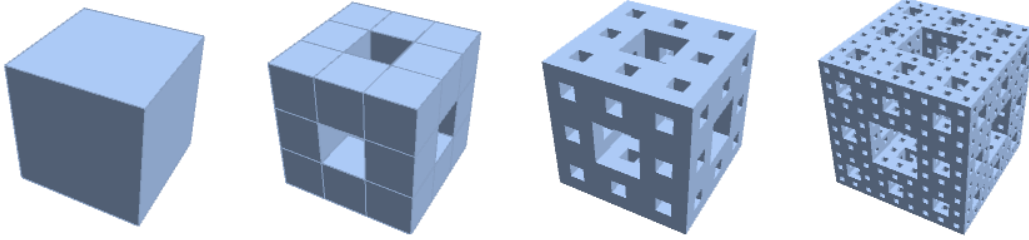3. $\mathbb{P}_p(X_{i_{1_1}\ldots i_{1_n},\ldots,i_{d_1}\ldots i_{d_n}}) = p_{i_{1_n},\ldots,i_{d_n}}$.

Figure 1.2: The zeroth, first, second and third level approximation of the Menger sponge.

Thus

$$\mathcal{E}_n = \{i_{1_1} \ldots i_{1_n}, \ldots, i_{d_1} \ldots i_{d_n} : X_{i_{1_1}, \ldots, i_{d_1}} = X_{i_{1_1} i_{1_2}, \ldots, i_{d_1} i_{d_2}} = \ldots$$
$$= X_{i_{1_1} \ldots i_{1_n}, \ldots, i_{d_1} \ldots i_{d_n}} = 1\}.$$

Now the nth level approximation of Λ is Λ_n, defined by the survival set $\mathcal{E}_n$:

$$\Lambda_n = \bigcup_{(\underline{i}_{1_n}, \ldots, \underline{i}_{d_n}) \in \mathcal{E}_n} I_{\underline{i}_{1_n}, \underline{i}_{d_n}} \text{ and from that } \Lambda = \bigcap_{n=1}^{\infty} \Lambda_n. \qquad (1.3)$$

The above defined Λ is random variable i.e. $\Lambda \colon \Omega \to \{\text{the Cantor sets of } I^d\}$, where Ω is an infinite randomly labeled tree, defined above.

Definition 1.2. *We say that the Mandelbrot percolation is homogeneous if all the probabilities are the same i.e. $p_{i_1, \ldots, i_d} = p$ for all $i_1, \ldots, i_d \in \{0, \ldots, M-1\}$ for some $p \in [0, 1]$. Otherwise we say that it is inhomogeneous.*

1.2 Menger sponge

Definition 1.3 (Menger sponge). *The Menger sponge is the attractor of the following iterated function system:*

$$\mathcal{S} = \left\{ S_{i,j,k}(\underline{x}) = \frac{1}{3}(\underline{x} + (i,j,k)) \right\}_{(i,j,k) \in \mathcal{J}},$$

where

$$\mathcal{J} = \{0,1,2\}^3 \setminus \{(1,1,0),(1,0,1),(0,1,1),(1,1,2),(1,2,1),(2,1,1),(1,1,1)\}.$$

$$(1.4)$$

Definition 1.4. *The (homogeneous) random Menger sponge is a three-dimensional Mandelbrot percolation, with probabilities $p_{i,j,k} = p$ for $(i,j,k) \in \mathcal{J}$ and $p_{i,j,k} = 0$ for $(i,j,k) \notin \mathcal{J}$ for some $p \in [0,1]$.*

We denote the random Menger sponge with $\mathcal{M}_p$.

Chapter 2

Interval in the projection

2.1 Theorem and Proof

Let $I = [0,1]^3$ and $I_{\underline{l}_n, \underline{m}_n, \underline{j}_n}$ as in (1.2) Let $M = 3$ and $\Lambda = \Lambda_{\underline{p}}^M$ is the three dimensional Mandelbrot percolation with vector of probabilities $\underline{p} = \{p_{0,0,0}, \ldots, p_{2,2,2}\} \in [0,1]^{27}$, and let Λ_n denote the nth level approximation of Λ as in (1.3). Let $S_{l,m,j} : \mathbb{R}^3 \to \mathbb{R}^3$ for $(l,m,j) \in \{0,1,2\}^3$ defined as:

$$S_{l,m,j}(x,y,z) = \frac{1}{3}[(x,y,z) + (l,m,j)]. \tag{2.1}$$

And let

$$S_{\underline{l}_n, \underline{m}_n, \underline{j}_n}(\underline{x}) = S_{l_n, m_n, j_n} \circ \cdots \circ S_{l_1, m_1, j_1}(\underline{x}).$$

Hence $I_{l,m,j} = S_{l,m,j}(I)$ and similarly $I_{\underline{l}_n, \underline{m}_n, \underline{j}_n} = S_{\underline{l}_n, \underline{m}_n, \underline{j}_n}(I)$. Let

$$\gamma_k = \sum_{l+m+n \equiv k \pmod 3} p_{l,m,n} \text{ for } k \in \{0,1,2\}. \tag{2.2}$$

Now we define shape A, B and C, later called level 0 shapes. Shape A is the tetrahedron defined by $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$, B is the tetrahedron
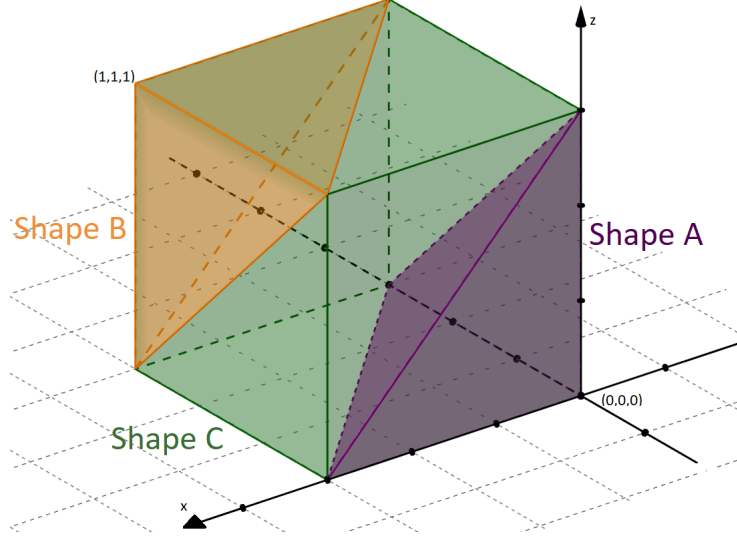
Figure 2.1: Shape A, B and C.

defined by $(1,1,1),(1,1,0),(0,1,1),(1,0,1)$ and C is $[0,1]^3 \setminus (A \cup B)$. Define $\mathcal{C}_i = \{(x,y,z) : x + y + z \in [i, i+1]\}$, for $i = 0, 1, 2$ and

$$\mathcal{C}_{\underline{i}_n} = \mathcal{C}_{i_1,\dots,i_n} = \left\{ (x,y,z) : x + y + z \in \left[\sum_{k=0}^{n} i_k \cdot \frac{1}{M^k}, \sum_{k=0}^{n} i_k \cdot \frac{1}{M^k} + \frac{1}{M^n} \right] \right\}$$

which we call a level $n-1$ column, note that $A = [0,1]^3 \cap \mathcal{C}_0$, $B = [0,1]^3 \cap \mathcal{C}_2$, $C = [0,1]^3 \cap \mathcal{C}_1$. We define the shapes of subcubes of I, indexed as the subcubes, called level n shapes:

$$A_{\underline{l}_n, \underline{m}_n, \underline{j}_n} := S_{\underline{l}_n, \underline{m}_n, \underline{j}_n}(A)$$

$$C_{\underline{l}_n, \underline{m}_n, \underline{j}_n} := S_{\underline{l}_n, \underline{m}_n, \underline{j}_n}(C)$$

$$B_{\underline{l}_n, \underline{m}_n, \underline{j}_n} := S_{\underline{l}_n, \underline{m}_n, \underline{j}_n}(B)$$

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Let $Z^{U,V}(k)$ denote the number of level 1 V shapes in a level 0 U shape in the level 1 column k:

$$Z^{U,V}(k) = \#\{(l,m,n) \in \mathcal{E}_1 : V_{l,m,n} \subset \mathcal{C}_{i,k},$$

$$\text{where } i = 0 \text{ if } U = A, i = 1 \text{ if } U = C, i = 2 \text{ if } U = B\}.$$

Note that $\mathcal{E}_n$ was defined in Definition 1.1, it denotes the indices of the retained level n cubes. Similarly $Z^{U,V}(\underline{k_n})$ is the number of level n (retained) V shape in $\mathcal{C}_{i,\underline{k_n}}$, $i = 0$ if $U = A$, $i = 1$ if $U = C$, $i = 2$ if $U = B$. Denote

$$Z^V(k) = \sum_{U \in \{A,B,C\}} Z^{U,V}(k), \tag{2.3}$$

and similarly $Z^V(\underline{k_n}) = \sum_{U \in \{A,B,C\}} Z^{U,V}(\underline{k_n})$ For $k \in \{0,1,2\}$ let $M(k)$ denote the expectation matrices, namely:

$$M(k) = \begin{bmatrix} \mathbb{E}(Z^{A,A}(k)) & \mathbb{E}(Z^{A,B}(k)) & \mathbb{E}(Z^{A,C}(k)) \\ \mathbb{E}(Z^{B,A}(k)) & \mathbb{E}(Z^{B,B}(k)) & \mathbb{E}(Z^{B,C}(k)) \\ \mathbb{E}(Z^{C,A}(k)) & \mathbb{E}(Z^{C,B}(k)) & \mathbb{E}(Z^{C,C}(k)) \end{bmatrix}$$

and for $\underline{k_n} \in \{0,1,2\}^n$

$$M(\underline{k_n}) = \begin{bmatrix} \mathbb{E}(Z^{A,A}(\underline{k_n})) & \mathbb{E}(Z^{A,B}(\underline{k_n})) & \mathbb{E}(Z^{A,C}(\underline{k_n})) \\ \mathbb{E}(Z^{B,A}(\underline{k_n})) & \mathbb{E}(Z^{B,B}(\underline{k_n})) & \mathbb{E}(Z^{B,C}(\underline{k_n})) \\ \mathbb{E}(Z^{C,A}(\underline{k_n})) & \mathbb{E}(Z^{C,B}(\underline{k_n})) & \mathbb{E}(Z^{C,C}(\underline{k_n})) \end{bmatrix}.$$

Lemma 2.1. *For any* $\underline{k_n} \in \{0,1,2\}^n$, $\underline{k_n} = (k_1, \ldots, k_n)$: $M(\underline{k_n}) = M(k_1) \ldots M(k_n)$

Proof. First we prove it for $\underline{k_2} = (k_1, k_2)$. By the construction if $W_{l_1,m_1,n_1} \subset \mathcal{C}_{i,k_1}$, then $V_{l_1 l_2, m_1 m_2, n_1 n_2} \subset \mathcal{C}_{i,k_1,k_2} \cap W_{l_1,m_1,n_1}$ if and only if $V_{l_2,m_2,n_2} \subset \mathcal{C}_{j,k_2}$, $j = 0$ if $W = A$, $j = 1$ if $W = C$, $j = 2$ if $W = B$, for any $i, j, l_1, l_2, m_1, m_2, n_1, n_2 \in$

$\{0, 1, 2\}$, $V, W \in \{A, B, C\}$. Fix $U, V \in \{A, B, C\}$ and let $i = 0$ if $U = A$, $i = 1$ if $U = C$, $i = 2$ if $U = B$.

$$\mathbb{E}(Z^{U,V}(k_1 k_2))$$

$$= \mathbb{E}\Big(\sum_{\substack{W \\ \in \{A,B,C\}}} \sum_{\substack{W_{l_1,m_1,n_1} \\ \subset \mathcal{C}_{i,k_1}}} \mathbb{1}[(l_1, m_1, n_1) \in \mathcal{E}_1] \sum_{\substack{V_{\underline{l_2},\underline{m_2},\underline{n_2}} \subset \mathcal{C}_{i,k_1 k_2} \\ V_{\underline{l_2},\underline{m_2},\underline{n_2}} \subset W_{l_1,m_1,n_1}}} \mathbb{1}[(\underline{l_2}, \underline{m_2}, \underline{n_2}) \in \mathcal{E}_2] \Big)$$

$$= \sum_{\substack{W \\ \in \{A,B,C\}}} \sum_{\substack{W_{l_1,m_1,n_1} \\ \subset \mathcal{C}_{i,k_1}}} \mathbb{P}((l_1, m_1, n_1) \in \mathcal{E}_1) \sum_{\substack{V_{\underline{l_2},\underline{m_2},\underline{n_2}} \subset \mathcal{C}_{i,k_1 k_2} \\ V_{\underline{l_2},\underline{m_2},\underline{n_2}} \subset W_{l_1,m_1,n_1}}} \mathbb{P}((\underline{l_2}, \underline{m_2}, \underline{n_2}) \in \mathcal{E}_2 | (l_1, m_1, n_1) \in \mathcal{E}_1)$$

$$= \sum_{\substack{W \\ \in \{A,B,C\}}} \sum_{\substack{W_{l_1,m_1,n_1} \\ \subset \mathcal{C}_{i,k_1}}} p_{l_1,m_1,n_1} \sum_{\substack{V_{\underline{l_2},\underline{m_2},\underline{n_2}} \subset \mathcal{C}_{i,k_1 k_2} \\ V_{\underline{l_2},\underline{m_2},\underline{n_2}} \subset W_{l_1,m_1,n_1}}} p_{l_2,m_2,n_2}$$

$$= \sum_{\substack{W \\ \in \{A,B,C\}}} \sum_{\substack{W_{l_1,m_1,n_1} \\ \subset \mathcal{C}_{i,k_1}}} p_{l_1,m_1,n_1} \mathbb{E}(Z^{W,V}(k_2))$$

$$= \sum_{W \in \{A,B,C\}} \mathbb{E}(Z^{U,W}(k_1)) \mathbb{E}(Z^{W,V}(k_2))$$

This proves that $M(k_1 k_2) = M(k_1) M(k_2)$. Now assume that for $s - 1$: $M(\underline{k}_{s-1}) = M(k_1) \dots M(k_{s-1})$, then by a similar argument to the above:

$$\mathbb{E}(Z^{U,V}(\underline{k}_s)) = \sum_{W \in \{A,B,C\}} Z^{U,W}(\underline{k}_{s-1}) Z^{W,V}(k_s)$$

Which shows, that $M(\underline{k}_s) = M(\underline{k}_{s-1}) M(k_s)$, which by the induction hypothesis equals $M(k_1) \dots M(k_s)$. $\qquad\square$

In the case of the three dimensional inhomogeneous Mandelbrot percolation with M=3, the level one expectation matrices are the following:

$$M(0) = \begin{bmatrix} u_0 & 0 & 0 \\ u_6 & u_4 & u_5 \\ u_3 & u_1 & u_2 \end{bmatrix}, M(1) = \begin{bmatrix} u_1 & 0 & u_0 \\ 0 & u_5 & u_6 \\ u_4 & u_2 & u_3 \end{bmatrix}, M(2) = \begin{bmatrix} u_2 & u_0 & u_1 \\ 0 & u_6 & 0 \\ u_4 & u_3 & u_4 \end{bmatrix}.$$

$$(2.4)$$

Where

$$u_i = \sum_{k+l+m=i} p_{k,l,m}.$$

For example $u_0 = p_{0,0,0}$ and $u_5 = p_{1,2,2} + p_{2,1,2} + p_{2,2,1}$. Hence the column sums of the matrices are equal to γ_i for some i, the column sums of $M(0)$ are $\gamma_0, \gamma_1, \gamma_2$ respectively, the column sums of $M(1)$ are $\gamma_1, \gamma_2, \gamma_0$ and lastly, the column sums of $M(2)$ are $\gamma_2, \gamma_0, \gamma_1$. Let

$$\gamma := \min\{\gamma_0, \gamma_1, \gamma_2\}.$$

Condition 1. $\gamma > 1$.

Remark. *Under Condition 1 the Hausdorff dimension of Λ is greater than 1 almost surely conditioned on non extinction, because it is proved in [5] that under non extinction a.s.*

$$\dim_H(\Lambda) = \frac{\log(\sum_{(i,j,k)\in\{0,1,2\}^3} p_{i,j,k})}{\log(3)} = \frac{\log(\gamma_0 + \gamma_1 + \gamma_2)}{\log(3)} > \frac{\log(3)}{\log(3)} = 1.$$

Condition 2. There exists a $k \in \{0, 1, 2\}$ such that at least one of the rows of $M(k)$ is strictly positive.

Let *proj* denote the orthogonal projection to the line $\{t\underline{e} \mid t \in \mathbb{R}\}$, where $\underline{e} = (1, 1, 1)$.

Theorem 2.2. *Assume that Conditions 1, 2 hold. Then the orthogonal $proj(\Lambda)$ contains an interval almost surely conditioned on Λ being non empty.*

The intuitive meaning of these conditions are the following: we will later define a process that starts from a triplet of different shapes, and we count the shape triplets in every column in every level coming from the first triplet. Condition 2 guarantees that we can start the process with positive probability

(we can find a column in which we keep at least one of each shape with positive probability) and Condition 1 guarantees that the process does not die out with positive probability. The main steps of the proof are the following:

- First we show that we can find a level 1 column where with positive probability all types of shapes are retained, this means that we can start the process.

- Then we will show that starting from this column with positive probability in every level in every subcolumn of the column we find at least one of all the shapes A, B and C. This means that with positive probability the projection of the cube contains an interval.

- After that, using statistical self-similarity, and that all the level n cubes are conditionally independent and that we retain exponential number of level n cubes conditioned on non-extinction (because the dimension of the set is larger than 1), we will show that the projection contains an interval a.s. conditioned on Λ being non empty.

Lemma 2.3. *Assume that Condition 2 holds. Then there exist* $i, j \in \{0, 1, 2\}$ *such that* $\mathcal{C}_{i,j} \cap \Lambda_1$ *contains at least one from each of the level one shapes A,B and C with positive probability.*

Proof. From Condition 2 we know that there exists a j and an i such that $\underline{e}_i^T M(j) > 0$. Let $X_U := \#\{(k, l, m) \in \mathcal{E}_1 : U_{k,l,m} \subset \mathcal{C}_{i,j}\}$ for $U \in \{A, B, C\}$, that is X_U counts the retained U shapes in $\mathcal{C}_{\rangle,|}$. $\mathbb{P}(X_A > 0$ and $X_B > 0$ and $X_C > 0) > 0$ if and only if $\mathbb{E}(X_A \cdot X_B \cdot X_C) > 0$. The random variables X_A, X_B and X_C are independent, because they are counting shapes in the
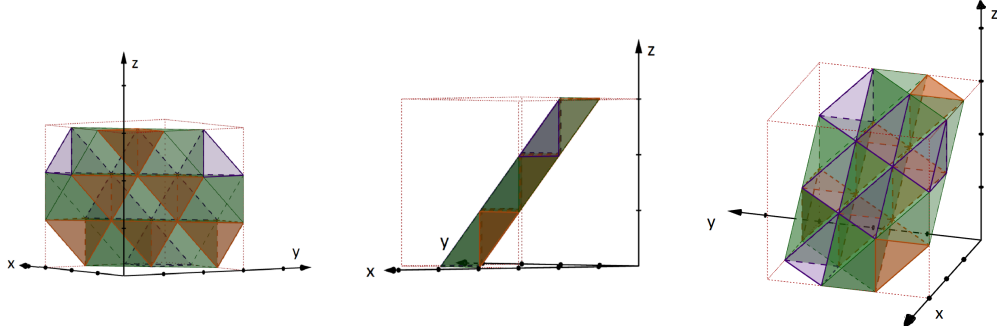
Figure 2.2: The shapes of $\mathcal{C}_{1,1}$ in case of $\underline{p} = \underline{1}$ from different angles.(Purple, yellow an green denotes shape A, B and C respectively.)

same column, all the shapes are coming from different level one cubes, which are independent, hence

$$\mathbb{E}(X_A \cdot X_B \cdot X_C) = \mathbb{E}(X_A) \cdot \mathbb{E}(X_B) \cdot \mathbb{E}(X_C)$$
$$= \underline{e}_i^T M(j)\underline{e}_1 \cdot \underline{e}_i^T M(j)\underline{e}_2 \cdot \underline{e}_i^T M(j)\underline{e}_3 > 0.$$

The last inequality follows from the fast that $\underline{e}_i^T M(j) > 0$. $\qquad\square$

Denote

$$p_0 := \mathbb{P}(X_A > 0 \text{ and } X_B > 0 \text{ and } X_C > 0) > 0. \qquad (2.5)$$

Fact 2.4. *For the non negative $m \times m$ matrices A and B*

$$\min_i \underline{e}^T (A \cdot B)\underline{e}_i \geq \min_i \underline{e}^T (A)\underline{e}_i \cdot \min_i \underline{e}^T (B)\underline{e}_i$$

where $\underline{e} = (1, \ldots, 1) \in \mathbb{R}^m$, and $\underline{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^m$ with a 1 at the ith position.

It follows from the Fact above, that the column sums of $M(\underline{k}_n)$ are greater or equal to γ^n, as the column sums of $M(k)$ $k \in \{0, 1, 2\}$ are greater or equal than γ.

Lemma 2.5. *Assume that Condition 1 holds. Then for any n :*

$$\mathbb{P}(Z^A(\underline{k}_n) \geq \gamma^n \ and \ Z^B(\underline{k}_n) \geq \gamma^n \ and \ Z^C(\underline{k}_n) \geq \gamma^n, \ \forall \underline{k}_n \in \{0,1,2\}^n) > 0$$

Proof. This proof is an adaptation of that of Dekking and Simon [4]. We would like to count the number of level n A and B and C shapes in columns $\mathcal{C}_{0,\underline{k}_n}, \mathcal{C}_{1,\underline{k}_n}, \mathcal{C}_{2,\underline{k}_n}$ together that are retained with positive probability. To do this we consider the deterministic case, when $\hat{p}_{i,j,k} = 0$ if $p_{i,j,k} = 0$ and 1 otherwise. Let $\hat{M}(k)$ denote the expectation matrix with respect to the probability vector $\hat{\underline{p}}$. Then the column sums of $\hat{M}(\underline{k}_n)$ counts the shapes in $\cup_{i \in \{0,1,2\}} \mathcal{C}_{i,\underline{k}_n}$ that we keep with positive probability.

$$\underline{e}^T \hat{M}(k_n) \geq \underline{e}^T M(k_n) \geq \gamma^n \cdot \underline{e}^T$$

Hence for one $\underline{k}_n$ the probability is positive. The events for different $\underline{k}_n$s are not independent, but they are not mutually exclusive, hence their intersection also has positive probability. □

Without loss of generality we may assume that in Lemma 2.3, $i = j = 1$. Let

$$N(\underline{k}_n) = \min_{U \in \{A,B,C\}} \{Z^U(11\underline{k}_n)\}$$

, and A_n be the event that this minimum grows exponentially in n for all $\underline{k}_n$, namely

$$A_n = \{N(\underline{k}_n) \geq \eta^n, \quad \forall \underline{k}_n \in \{0,1,2\}^n\}$$

where $1 < \eta < \min\{2, \gamma\}$. By Lemma 2.5 and Lemma 2.3, $\mathbb{P}(A_n) > 0$ for all n.

First we prove that

$$\mathbb{P}(A_n \text{ holds for all } n \geq r) \geq \mathbb{P}(A_r) \prod_{k=r+1}^{\infty} (1 - 3^{k+2} \delta^{\eta^k}) \text{ for some } 0 < \delta < 1.$$

And from that $\mathbb{P}(N(\underline{k}_n) > 0, \quad \forall \underline{k}_n \in \{0,1,2\}^n, \quad \forall n) > 0$ follows, with the right choice of r.

From the Azuma-Hoeffding inequality it follows that for independent $Z_1^U(k), \ldots, Z_m^U(k); Z_1^U(k) \sim Z^U(k)$:

$$\mathbb{P}(Z_1^U(k) + \cdots + Z_m^U(k) \leq m\eta) \leq \delta^m \text{ for some } 0 < \delta < 1. \qquad (2.6)$$

We will use this large deviation bound to give an upper bound on $\mathbb{P}(\overline{A_{n+1}}|A_n)$, where $\overline{A_{n+1}}$ is the complement of the event A_{n+1}, as usual. For this we would like to have a similar situation as in (2.6). We start with a given level n+1 column $\mathcal{C}_{1,1,\underline{k}_n}$, and then we will use the union bound multiple times to be able to use (2.6) and later to upper bound $\mathbb{P}(\overline{A_{n+1}}|A_n)$ we again use the union bound.

$$\mathbb{P}(\exists k \text{ s.t. } N(\underline{k}_n k) < \eta^{n+1} \mid N(\underline{k}_n) \geq \eta^n)$$
$$= \mathbb{P}(N(\underline{k}_n 0) < \eta^{n+1} \text{ or } N(\underline{k}_n 0) < \eta^{n+1} \text{ or } N(\underline{k}_n 0) < \eta^{n+1} \mid N(\underline{k}_n) \geq \eta^n)$$
$$\leq \sum_{k=0}^{2} \mathbb{P}(N(\underline{k}_n k) < \eta^{n+1} \mid N(\underline{k}_n) \geq \eta^n)$$

$$\mathbb{P}(N(\underline{k}_n k) < \eta^{n+1} \mid N(\underline{k}_n) \geq \eta^n)$$
$$= \mathbb{P}(Z^A(11\underline{k}_n k) < \eta^{n+1} \text{ or } Z^B(11\underline{k}_n k) < \eta^{n+1} \text{ or } Z^C(11\underline{k}_n k) < \eta^{n+1} \mid N(\underline{k}_n) \geq \eta^n)$$
$$\leq \sum_{U \in \{A,B,C\}} \mathbb{P}(Z^U(11\underline{k}_n k) \leq \eta^{n+1} \mid N(\underline{k}_n) \geq \eta^n).$$

For the next step we show that we can use large deviation theory. As $Z^U(\underline{k}_n k)$ is the number of U shapes in $\mathcal{C}(11\underline{k}_n k)$ and we conditioned the event on $N(\underline{k}_n) \geq \eta^n$, we know that:

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- In $\mathcal{C}(\underline{k}_n)$ we have at least η^n of every shape, i.e. we have at least η^n ABC triplets.

- Coming from A, B and C together the expected number of retained shapes of a given shape in the next level is greater than η for all shapes.

- In the ABC triplets the number of retained U shapes in a column k has the same distribution as $Z^U(k)$, and what happens in different shapes and triplets are independent for a given k.

Thus for any $U \in \{A, B, C\}$ using (2.6):

$$\mathbb{P}(Z^U(11\underline{k}_n k) \leq \eta^{n+1} \mid N(\underline{k}_n) \geq \eta^n) \leq \mathbb{P}(Z_1^U(k) + \cdots + Z_{\eta^n}^U(k) \leq \eta\eta^n) \leq \delta^{\eta^n}.$$

Hence

$$\mathbb{P}(\exists k \text{ s.t. } N(\underline{k}_n k) \leq \eta^{n+1} \mid N(\underline{k}_n) \geq \eta^n)$$
$$\leq \sum_U \sum_k \mathbb{P}(Z^U(11\underline{k}_n k) < \eta^{n+1} \mid N(\underline{k}_n) \geq \eta^n) \leq 3^2 \delta^{\eta^n}.$$

hence

$$\mathbb{P}(\overline{A_{n+1}} \mid A_n) = \mathbb{P}\left(\bigcup_{\substack{\underline{k}_n \\ \in \{0,1,2\}^n}} \{\exists k \text{ s.t. } N(\underline{k}_n k) \leq \eta^{n+1}\} \mid \bigcap_{\substack{\underline{k}_n \\ \in \{0,1,2\}^n}} \{N(\underline{k}_n) \geq \eta^n\} \right)$$

And because $\#\{0, 1, 2\}^n = 3^n$, using the union bound:

$$\mathbb{P}(\overline{A_{n+1}} \mid A_n) \leq 3^{n+2} \delta^{\eta^n},$$

therefore

$$\mathbb{P}(A_{n+1} \mid A_n) \geq (1 - 3^{n+2} \delta^{\eta^n})$$

and

$$\mathbb{P}(A_r \cap \cdots \cap A_n) \geq \mathbb{P}(A_r) \prod_{i=r}^{n} (1 - 3^{n+2} \delta^{\eta^n}).$$

As $\mathbb{P}(A_r) > 0 \; \forall r$, we can choose r in a way, that $\prod_{i=r}^{n}(1 - 3^{i+2}\delta^{\eta^i}) > 0$.

This means that with positive probability the projection of $I \cap \Lambda$ contains an interval, denote this probability with θ. Thus for a level n cube $I_{\underline{l}_n, \underline{m}_n, \underline{j}_n}$ the probability that $I_{\underline{l}_n, \underline{m}_n, \underline{j}_n} \cap \Lambda$ contains an interval conditioned on $I_{\underline{l}_n, \underline{m}_n, \underline{j}_n} \subset \Lambda^n$ is θ, because the percolation starting from $I_{\underline{l}_n, \underline{m}_n, \underline{j}_n}$ (conditioned on the event that the cube is retained) has the same distribution as the original Mandelbrot percolation. Using the fact that the number of level n cubes $\#\mathcal{E}_n$ tends to infinity as n tends to infinity conditioned on Λ being non empty, and that the processes runs independently in every level-n retained cube, we can conclude that the projection of Λ contains an interval a.s. conditioned on Λ being non empty, because the projection contains no interval if and only if for every retained level n cube the projection of the intersection with Λ contains no interval.Let $\text{Int}(A)$ denote the interior of the set A, then by the above reasoning:

$$\mathbb{P}(\text{Int}(proj(\Lambda)) = \emptyset \mid \Lambda \neq \emptyset) \leq \mathbb{P}(\#\mathcal{E}_n < N \mid \Lambda \neq \emptyset) + (1 - \theta)^N,$$

then let $n \to \infty$ and $N \to \infty$ gives the desired result.

2.2 The existence of intervals in the projection of $\mathcal{M}_p$

In this subsection first we verify that for $p > \frac{1}{6}$ the projection $proj$ of the random Menger sponge $\mathcal{M}_p$ contains an interval. Then we prove that for

$p < \frac{1}{6}$ the projection *proj* of the random Menger sponge does not contain any intervals.

For the random Menger sponge $\mathcal{M}_p$ (see Definition 1.4) the expectation matrices (for the general case see (2.4)) are the following:

$$M(0) = \begin{bmatrix} p & 0 & 0 \\ p & 3p & 3p \\ 6p & 3p & 3p \end{bmatrix}, M(1) = \begin{bmatrix} 3p & 0 & p \\ 0 & 3p & p \\ 3p & 3p & 6p \end{bmatrix}, M(2) = \begin{bmatrix} 3p & p & 3p \\ 0 & p & 0 \\ 3p & 6p & 3p \end{bmatrix}.$$

(2.7)

First we prove that for $p > \frac{1}{6}$ $\text{Int}(proj(\mathcal{M}_p))$ is not empty almost surely conditioned on non-extinction. As for $i \in \{0, 1, 2\}$ γ_i denotes the column sums of the expectation matrices, we can see that $\gamma_0 = 8p$ and $\gamma_1 = \gamma_2 = 6p$. For $p > 1/6$ Condition 1 holds, because in that case $\gamma = \min_{i \in \{0,1,2\}} \gamma_i > 1$, and Condition 2 also holds, because for example the third row of $M(1)$ is strictly positive, hence for $p > 1/6$ the projection of the random Menger sponge contains an interval almost surely conditioned on non-extinction.

Now assume $p < \frac{1}{6}$, we will prove that the projection does not contain an interval almost surely following the ideas of [4]. The eigenvalues of M(0) are $\lambda_1(M(0)) = 0$, $\lambda_2(M(0)) = p$, $\lambda_3(M(0)) = 6p$. M(0) and M(2) are similar matrices, hence they have the same eigenvalues. $p < \frac{1}{6}$, hence the spectral radius of M(0) (and also M(2)) is smaller than 1, which means that the powers of M(0) and M(2) tends to 0. Hence for any matrix norm (therefore specially for the 1-norm) $\lim_{n\to\infty} ||M(0)|| = \lim_{n\to\infty} ||M(2)|| = 0$. Thus, for any n, for any $(k_1, \ldots, k_n) \in \{0, 1, 2\}^n$, by Lemma 2.1 and the submultiplicativity

of the 1-norm:

$$\lim_{j\to\infty} ||M(k_1\ldots k_n \underbrace{000\ldots0}_{j \text{ times}})||_1 = \lim_{j\to\infty} ||M(k_1\ldots k_n)\,M(\underbrace{000\ldots0}_{j \text{ times}})||_1 \le$$

$$\lim_{j\to\infty} ||M(k_1\ldots k_n)||_1 ||M(\underbrace{000\ldots0}_{j \text{ times}})||_1 = \underbrace{||M(k_1\ldots k_n)||_1}_{<\infty} \lim_{j\to\infty} ||M(\underbrace{000\ldots0}_{j \text{ times}})||_1$$

$$= 0.$$

For any interval $J \subset \{t\underline{e} \mid t \in \mathbb{R}\} \cap [0,1]^3$ we can find an n and $k_1,\ldots,k_n$ such that $J \subset proj(\mathcal{C}(k_1,\ldots,k_n))$. We will show that for this $\mathcal{C}(k_1,\ldots,k_n)$:

$$\bigcap_{m=0}^{\infty} proj(\mathcal{C}(k_1,\ldots,k_n,\underbrace{0,\ldots,0}_{m \text{ times}}))$$

is not contained in the projection almost surely. That is let Z_m denote the number of retained shapes in $\mathcal{C}(k_1,\ldots,k_n,\underbrace{0,\ldots,0}_{m \text{ times}})$, then Z_m is the sum of the number of shapes A, B and C in $\mathcal{C}(k_1,\ldots,k_n,\underbrace{0,\ldots,0}_{m \text{ times}})$. The expected number of these are the column sums of $M(k_1\ldots k_n \underbrace{000\ldots0}_{m \text{ times}})$ respectively (the first column sum is the expected number of shapes A, and so on), hence the expectation of the sum of shapes which is the sum of the expectation of the shapes, is smaller than 3 times the maximum number of shapes, i.e. the 1 norm of $M(k_1\ldots k_n \underbrace{000\ldots0}_{m \text{ times}})$:

$$\mathbb{E}(Z_m) \le 3||M(k_1\ldots k_n \underbrace{000\ldots0}_{m \text{ times}})||_1$$

and by Markov's inequality

$$\mathbb{P}(Z_m \ge 1) \le \mathbb{E}(Z_m) \to 0 \text{ as } m \to \infty.$$

The projection has zero Lebesgue measure a.s.

The projection does not contain an interval a.s.

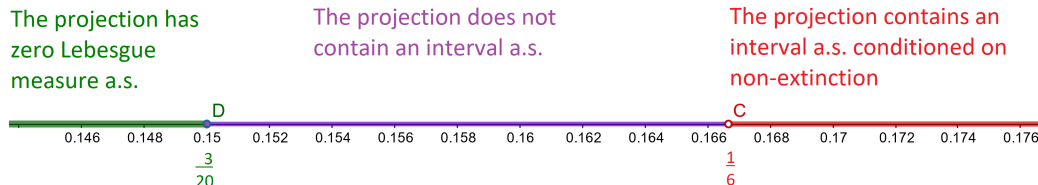The projection contains an interval a.s. conditioned on non-extinction

Figure 2.3: The parameter intervals for p in the case of the random Menger sponge.

Hence for one given interval the probability that this interval is contained in the projection is 0. To prove that this hold simultaneously for all the intervals of $\{t\underline{e}|t \in \mathbb{R}\} \cap [0,1]^3$, observe that for any J interval of $\{t\underline{e}|t \in \mathbb{R}\} \cap [0,1]^3$ we can find a sub-interval $J_{a,b}$ with rational endpoints a and b such that $J_{a,b} \subset J$, hence if the projection contains J than it also contains $J_{a,b}$. Since there are only countably many rational numbers we can use the union bound to prove the statement.

$\mathbb{P}(\exists \text{ an interval } J \subset proj(\Lambda))$

$$\leq \mathbb{P}\left(\bigcup_{\substack{a,b\in proj(I)\\a,b\in\mathbb{Q}\\a<b}} J_{a,b} \subset proj(\Lambda)\right) \leq \sum_{b}\sum_{a}\mathbb{P}(J_{a,b} \subset proj(\Lambda)) = 0.$$

We know that the three-dimensional homogeneous Mandelbrot percolation contains an interval a.s. conditioned on non extinction in the case when it's Hausdorff dimension is greater one. We can see that this is not the case here because conditioned on non extinction the random Menger sponge has Hausdorff dimension $\dim_H(\mathcal{M}_p) = \frac{\log(20p)}{\log(3)}$ with probability 1, which is greater than 1 iff $p > \frac{3}{20} = \frac{9}{60} < \frac{10}{60} = \frac{1}{6}$. In Figure 2.3 one can see our knowledge

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of the projection with different choice of the values of p at this point. When $p > \frac{1}{6}$, we are in the red interval, the projection contains an interval a.s. conditioned on non extinction. Whenever $p < \frac{1}{6}$ a.s. the projection does not contain an interval, at the purple interval we don't know more, but in the green interval, when $p < \frac{3}{20}$, we know that a.s. the projection of the Menger sponge has zero Lebesgue measure, since $\dim_H(\mathcal{M}_p) \geq \dim_H(proj(\mathcal{M}_p))$, hence the Hausdorff dimension of the projection is less than one, meaning it's Lebesgue measure is 0. In the next chapters we turn our attention to the purple interval in Figure 2.3.

Chapter 3

The Lebesgue measure of the projection

In this chapter we will give a condition under which the projection of the three-dimensional Mandelbrot percolation has positive Lebesgue measure almost surely conditioned on non extinction. The proof in this chapter follows the lines of [6], although we can not use the results of the article, because the argument is concentrated on the sum of two independent one-dimensional Mandelbrot percolations, but the proof can almost entirely be used in our case too. At some points we simplified the arguments since our situation is somewhat less complicated because of the lack of dependence, but the main ideas remained the same. In this chapter we will use the same notations as we did in the previous chapters.The main idea of the proof is to use the theory of Branching processes in random environments (in short, B.P.R.E.), hence we start with a brief overview of the relevant parts.

3.1 Branching processes in random environment

In this section we briefly summarize the relevant parts of [1]. Assume that θ_t is a discrete time stochastic process on some probability space where the set of elementary event is Θ, this will be the environmental process. For each $\theta \in \Theta$ there is an associated probability generating function: $\varphi_\theta(s) = \sum_j^\infty p_j(\theta)s^j$. For the realizations of $\bar{\theta} = (\theta_0, \theta_1, \dots)$ a branching process $\mathcal{Z}_n$ evolves, namely: let $\mathcal{Z}_0 = 1$, and $\mathcal{Z}_1 = \sum_{i=1}^{\mathcal{Z}_0} X_{1,i}$, where the random variables $X_{1,i}$ are independent and distributed according to the p.g.f. φ_{θ_0}, similarly: $\mathcal{Z}_2 = \sum_{i=1}^{Z_1} X_{2,i}$, where the random variables $X_{2,i}$ are independent and distributed according to φ_{θ_1}, and so on. We denote the probability and expectation corresponding to this branching process by Prob and $\mathcal{E}$. Assume that $\{\theta_i(\omega)\}_{i=1,2,\dots}$ is a stationary and ergodic process. Let B denote the event of extinction and we define the extinction probabilities $q, q(\bar{\theta})$ as follows:

$$B := \{\omega : \mathcal{Z}_n(\omega) = 0 \text{ for some n}\}$$

$$q := \text{Prob}(B)$$

$$q(\bar{\theta}) := \text{Prob}(B|\sigma(\theta_0, \theta_1, \dots)).$$

Further let $a^- = -\min(a, 0)$ and $a^+ = \max(a, 0)$. Then

Theorem 3.1 (Theorem 3 from [1]). *If*

$$\mathcal{E}[-\log(1 - \varphi_{\theta_0}(0))] < \infty \tag{3.1}$$

and

$$\mathcal{E}[\log(\varphi_{\theta_0}'(1))]^- < \mathcal{E}[\log(\varphi_{\theta_0}'(1))]^+ \leq \infty. \tag{3.2}$$

Then

$$\text{Prob}(q(\bar{\theta}) < 1) = 1.$$

3.2   Conditions, theorem and proof

Condition 3. $\Gamma = \gamma_0\gamma_1\gamma_2 > 1$ (for the definition of γ_i see (2.2))

Remark. *Assume that Condition 3 holds, then for*

$$\mathcal{D} := \{(i, j, k) \in \{0, 1, 2\}^3 : p_{i,j,k} > 0\}$$

we have $\#\mathcal{D} \geq 3$. Otherwise, it follows from (2.2) that at least one of γ_0, γ_1, γ_2 has to be 0. Observe that

$$\sum_{(i,j,k)\in\mathcal{D}} p_{i,j,k} \geq \#\mathcal{D} \sqrt[\#\mathcal{D}]{\prod_{(i,j,k)\in\mathcal{D}} p_{i,j,k}} > 3.$$

Namely, the first inequality follows from the inequality of arithmetic and geometric means and the second one follows Condition 3. Thus the a.s. value of the Hausdorff dimension under non-extinction:

$$\dim_H \Lambda = \frac{\log(\sum_{(i,j,k)\in\mathcal{D}} p_{i,j,k})}{\log(3)} > \frac{\log(3)}{\log(3)} = 1.$$

Condition 4. For all $i \in \{0, 1, 2\}$ the expectation matrix $M(i)$ has a positive row.

The geometric meaning of Condition 4 is that for every first level column $C_{.,k_1}$ we can find a 0-level shape U (in other words a zero level column $\mathcal{C}_i$) such that with positive probability at the next level we retain all three kind of shapes in C_{i,k_1} coming from U.

Theorem 3.2. *Assume that Conditions 3, 4 holds, than the projection of the three-dimensional Mandelbrot percolation, $proj(\Lambda)$, has positive Lebesgue measure almost surely conditioned on non-extinction.*

Proposition 3.3. *If Conditions 3,4 hold than there exist a Borel set $K \subset [0, \sqrt{3}]$ of positive measure such that for Lebesgue almost all $x \in K$ we have*

$$\mathbb{P}(x \in proj(\Lambda)) > 0.$$

First we verify Theorem 3.2 assuming Proposition 3.3. Namely, first we show that the projection has positive Lebesgue measure with positive probability if Proposition 3.3 holds, then using statistical self-similarity as in the proof of Theorem 2.2 we will show that the positive measure property is a 0-1 property, which completes the proof.

Proof of Theorem 3.2 assuming Proposition 3.3. First we will prove that with positive probability the Lebesgue measure of Λ is greater than 0. $\mathbb{E}(\mathcal{L}(proj(\Lambda)))$ ≥ 0 hence $\mathbb{E}(\mathcal{L}(proj(\Lambda))) > 0$ if and only if $\mathbb{P}(\mathcal{L}(proj(\Lambda)) > 0) > 0$. Hence it is enough to prove that $\mathbb{E}(\mathcal{L}(proj(\Lambda))) > 0$.

$$\mathbb{E}(\mathcal{L}(proj(\Lambda))) = \int_{\Omega} \mathcal{L}(proj(\Lambda(\omega))) \, d\mathbb{P}(\omega)$$

$$= \int_{\Omega} \int_{[0,\sqrt{3}]} \mathbb{1}[x \in proj(\Lambda(\omega))] \, dx \, d\mathbb{P}(\omega) \geq \int_{\Omega} \int_{K} \mathbb{1}[x \in proj(\Lambda(\omega))] \, dx \, d\mathbb{P}(\omega)$$

$$= \int_{K} \int_{\Omega} \mathbb{1}[x \in proj(\Lambda(\omega))] \, d\mathbb{P}(\omega) \, dx = \int_{K} \mathbb{P}(x \in proj(\Lambda)) dx > 0$$

Now we will prove that $\mathbb{P}(\mathcal{L}(proj(\Lambda)) > 0) = 1$ under non extinction. Let $\varepsilon := \mathbb{P}(\mathcal{L}(proj(\Lambda)) > 0) > 0$. For $(\underline{i}_m, \underline{j}_m, \underline{k}_m) \in \mathcal{E}_m$, the Mandelbrot percolations starting from $I_{\underline{i}_m, \underline{j}_m, \underline{k}_m}$ are realizations of independent copies of the original Mandelbrot percolation Λ, hence $\mathbb{P}(\mathcal{L}(proj(\Lambda)) = 0 \mid \mathcal{E}_n \geq N) \leq (1-\varepsilon)^N$.

Also we know that under non extinction, if the Hausdorff dimension is greater than 1, the number of retained squares grows exponentially, hence $\mathbb{P}(\mathcal{E}_n < N) \to 0$ as $n \to \infty$ for any N.

$$\mathbb{P}(\mathcal{L}(proj(\Lambda)) = 0 | \Lambda \neq \emptyset) = \mathbb{P}(\mathcal{L}(proj(\Lambda)) = 0 | \mathcal{E}_n \geq N)\mathbb{P}(\mathcal{E}_n \geq N)$$
$$+ \mathbb{P}(\mathcal{L}(proj(\Lambda)) = 0 | \mathcal{E}_n < N)\mathbb{P}(\mathcal{E}_n < N) \leq (1 - \varepsilon)^N + \mathbb{P}(\mathcal{E}_n < N).$$

The second part tends to 0 as $n \to \infty$ as we argued above, and $\varepsilon > 0$ hence $(1 - \varepsilon)^N$ also tends to 0 as $N \to \infty$. Thus indeed $\mathbb{P}(\mathcal{L}(proj(\Lambda)) = 0 | \Lambda \neq \emptyset) = 0$. $\square$

Proof of Proposition 3.3. We can use Lemma 2.3, because Condition 4 implies Condition 2. Hence there is an $i, j \in \{0, 1, 2\}$ such that with positive p_0 probability $\mathcal{C}_{i,j}$ contains a level-1 A, B and C shape, and let E denote this event: $E := \{\exists$ a level-1 $A, B, C \subset \mathcal{C}_{i,j} \cap \Lambda_1\}$. Without loss of generality we may assume that $\mathcal{C}_{i,j} = \mathcal{C}_{1,1}$. Let

$$K = proj(\mathcal{C}_{1,1}) = \left[\frac{4\sqrt{3}}{9}, \frac{5\sqrt{3}}{9}\right],$$

and $\mathcal{U} \sim Uniform(K)$, and $Prob$ be the distribution of $\mathcal{U}$ and $\mathcal{E}$ be the corresponding expectation. Then $\mathcal{U}$ has a triadic expansion:

$$\mathcal{U} = \frac{4\sqrt{3}}{9} + \frac{\sqrt{3}}{9}\left(\sum_{j=1}^{\infty} \frac{i_j}{3^j}\right)$$

If we can prove, that

$$\text{Prob}(\mathbb{P}(\mathcal{U} \in proj(\Lambda)) > 0) = 1, \tag{3.3}$$

then beacuse $p_0 > 0$, and K is a Borel set of positive Lebesgue measure we can conclude that Proposition 3.3 holds.

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To prove (3.3) above, we will use the theory of branching processes with random environment. Namely we will define a B.P.R.E, the triadic decomposition will provide the environment, and the branching process will be the counting process of the $\{A, B, C\}$ triplets in the columns given by the environment. First we fix a large N to be defined later (see (3.8)). Conditioned on the event E fix A_0, B_0, C_0 such that A_0 is an A shape, B_0 is a B shape and C_0 is C shape, and they are contained in $\mathcal{C}_{1,1} \cap \Lambda_1$. Now we define Triplet_k inductively for a given $\underline{i} = (i_0, i_1, \dots) \in \{0, 1, 2\}^{\mathbb{N}}$:

$$\text{Triplet}_0 = \{(A_0, B_0, C_0)\}$$

and if we have $\text{Triplet}_{k-1} = \{(A_1^{k-1}, B_1^{k-1}, C_1^{k-1}), \dots, (A_m^{k-1}, B_m^{k-1}, C_m^{k-1})\}$, then Triplet_k will contain all the level $kN + 1$ triplets in $C_{1,1,i_0,\dots,i_{kN}} \cap \Lambda_{kN+1}$ conditioned on E, more precisely a triplet (A, B, C) is in Triplet_k if

1. A is of shape A, B is of shape B, C is of shape C.

2. A, B and C are level $kN + 1$ shapes.

3. None of A, B and C is contained in other triplet from Triplet_k.

4. A, B, C $\subset C_{1,1,i_0,\dots,i_{kN}} \cap \Lambda_{kN+1}$.

5. A,B and C are descendants of some shapes of one triplet in Triplet_{k-1}.

Then for $\bar{\theta} = (\theta_0, \theta_1, \dots)$, where $\theta_k = (i_{kN+1}, \dots, i_{(k+1)N})$, let

$$\mathcal{Z}_0(\bar{\theta}) = 1$$

$$\mathcal{Z}_k(\bar{\theta}) = \#\text{Triplet}_k$$

$\mathcal{Z}_n$ is B.P.R.V, because $\mathcal{Z}_{k+1}(\bar{\theta}) = \sum_{i=1}^{\mathcal{Z}_k(\bar{\theta})} X_{k,i}(\bar{\theta})$, where $X_{k,i}(\theta)$ is the number of triplets coming from the ith triplet of Triplet_k. The random variables
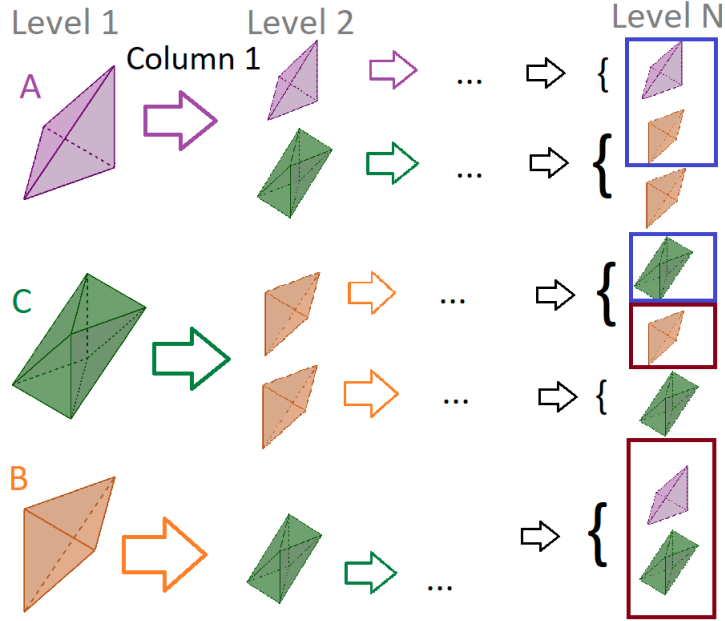
Figure 3.1: The explanation of the selection of the triplets: In this figure we show a possible realization. The initial triplet is A,B,C under the text "Level 1". In column 1 shape A gives birth to two shapes (this is denoted with the pink arrow), one is of type A (denoted with color pink) and on is of type C (denoted with color green). In the same column shape C gives birth to two shapes, both is of type B, and the last – the level-1 B shape gives birth to one shape of type C. The process goes on for the next N-2 levels, and the Nth level shapes coming from the second level shapes are below the text "Level N", for example the second level A shape N-2 levels later only has a descendant of shape A. The second level C shape, which comes from the first level A shape, has two descendant both of shape B, and so on. The shapes with blue border forms triplet, and the shapes with burgundy border forms another one, and we don't have any other triplets.

$X_{k,i}(\overline{\theta})$ are independent, because for a given m^{th} level column, and m^{th} level retained shapes U and V, what happens in U and V at later levels are independent of each other, since in a column the shapes are from different m^{th} level retained squares. Also they are identically distributed because they are independent realizations of scaled copies of the first triplet. $\theta_0, \theta_1, \ldots$ are independent and identically distributed, hence the environmental process is indeed a stationary ergodic process. Further if $\{\mathcal{Z}_n(\overline{\theta})\}$ does not die out for $\overline{\theta} = (\theta_0, \theta_1, \ldots) = (i_1, \ldots, i_N, \ldots, i_{kN+1}, \ldots, i_{(k+1)N}, \ldots)$, then conditioned on E:

$$\frac{4\sqrt{3}}{9} + \frac{\sqrt{3}}{9}\left(\sum_{j=1}^{\infty} \frac{i_j}{3^j}\right) \in proj(\Lambda).$$

Now we will use Theorem 3.1 to prove equation 3.3, but first we introduce some more notations. Let $q(k, V)$ denote the probability, that a level 0 shape V has descendant of at least one from each of the shapes in the kth column:

$q(k, V) = \mathbb{P}(\exists$ a level one A, B and C shape $\subset \mathcal{C}_{i,k} \cap \Lambda_1$,

where i=0 if V=A, i=1 if V=C and i=2 if V=B.).

Also

$$q = \min_{k \in \{0,1,2\}} \max_{V \in \{A,B,C\}} q(k, V).$$

Lemma 3.4. *Under Condition 4: $q > 0$.*

Proof. We prove that for any $k \in \{0, 1, 2\}$ $\max_{V \in \{A,B,C\}} q(k, V) > 0$. Fix k. Then by Condition 4 $M(k)$ has a strictly positive row, assume that it is the jth row and let U=A if j=1, U=B if j=2, and U=C if j=3. Let $X_W = \#\{$W is a retained level one W shape in U in the column k$\}$. We would like to prove that $\mathbb{P}(X_A > 0$ and $X_B > 0$ and $X_C > 0) > 0$, which is equivalent

to $\mathbb{E}(X_A \cdot X_B \cdot X_C) > 0$. As the events that we retain shapes in the same columns are independent $\mathbb{E}(X_A \cdot X_B \cdot X_C) = \mathbb{E}(X_A) \cdot \mathbb{E}(X_B) \cdot \mathbb{E}(X_C) = M(k)_{j,1} \cdot M(k)_{j,2} \cdot M(k)_{j,3} > 0.$ $\qquad\square$

Also let

$$U(k) = \arg\max_{V \in \{A,B,C\}} q(k,V) = \begin{cases} A & \text{if } \max_{V \in \{B,C\}} < q(k,A), \\ B & \text{if } \max_{V \in \{A,C\}} < q(k,B), \\ C & \text{otherwise,} \end{cases}$$

Now we examine the assumptions of Theorem 3.1, starting with (3.1).

$$\begin{aligned} \mathcal{E}[-\log(1-\varphi(0))] &= \frac{1}{3^N} \sum_{\substack{(i_1,\ldots,i_N) \\ \in \{0,1,2\}^N}} -\log(1-\varphi_{(i_1,\ldots,i_N)}(0)) \\ &= \frac{1}{3^N} \sum_{\substack{(i_1,\ldots,i_N) \\ \in \{0,1,2\}^N}} -\log(1-\varphi_{(i_1,\ldots,i_N)}(0)) \\ &= \frac{1}{3^N} \sum_{\substack{(i_1,\ldots,i_N) \\ \in \{0,1,2\}^N}} -\log(1-\mathbb{P}(\mathcal{Z}_1(\theta_0) = 0|\theta_0 = (i_1,\ldots,i_N)) \end{aligned}$$

That is if $\mathbb{P}(\mathcal{Z}_1(\theta_0) = 0|\theta_0 = (i_1,\ldots,i_N)) < 1$ for all $(i_1,\ldots,i_N) \in \{0,1,2\}^N$ then the above expression is less than infinity. By the definition of $q(k,V)$ and q:

$$\mathbb{P}(\mathcal{Z}_1(\theta_0) > 0|\theta_0 = (i_1,\ldots,i_N)) \geq p_0 q^N > 0,$$

hence the first assumption is satisfied. Take a look at the second assumption of Theorem 3.1, (3.2):

$$\mathcal{E}(\log(\varphi'_{\theta_0}(1))) = \frac{1}{3^N} \sum_{\substack{(i_1,\ldots,i_N) \\ \in \{0,1,2\}^N}} \log(\mathbb{E}(\mathcal{Z}_1(\bar{\theta})|\theta_0 = (i_1,\ldots,i_N)).$$

To verify that the second assumption also holds, we need to prove that

$$\frac{1}{3^N} \sum_{\substack{(i_1,\ldots,i_N) \\ \in \{0,1,2\}^N}} \log(\mathbb{E}(\mathcal{Z}_1(\bar{\theta})|\theta_0 = (i_1,\ldots,i_N)) > 0. \tag{3.4}$$

In order to do so recall the defintion of $Z^V(k)$ (equation (2.3)), using mathematical induction we show that for any $V \in \{A, B, C\}$:

$$\mathcal{E}[\log(\mathbb{E}(Z^V(\underline{i}_n)))] \geq n \log \sqrt[3]{\Gamma}, \tag{3.5}$$

where $\Gamma = \gamma_0 \cdot \gamma_1 \cdot \gamma_2$. For n=1, regardless of the choice of V:

$$\mathcal{E}[\log(\mathbb{E}(Z^V(i_0)))] = \frac{1}{3}(\log(\mathbb{E}(Z^V(0))) + \log(\mathbb{E}(Z^V(1))) + \log(\mathbb{E}(Z^V(2))))$$

$$= \frac{1}{3}(\log(\gamma_0) + \log(\gamma_1) + \log(\gamma_2)) = \log \sqrt[3]{\Gamma}.$$

Now assume that it holds for $n = k - 1$, we show that it holds for $n = k$. For a given $\underline{i}_k = (i_1,\ldots,i_k)$ and shape V, let $a_U = \mathbb{E}(Z^{U,V}(i_k))$, then $\sum_U a_U = \mathbb{E}(Z^V(i_k)) = \gamma_j$ for some j. It follows from Lemma 2.1, that

$$\mathbb{E}[Z^V(\underline{i}_k)] = \sum_{\substack{U \in \{A,B,C\} \\ a_U \neq 0}} a_U \mathbb{E}[Z^U(\underline{i}_{k-1})] = \sum_{\substack{U \in \{A,B,C\} \\ a_U \neq 0}} \frac{a_U}{\gamma_j}\gamma_j \mathbb{E}[Z^U(\underline{i}_{k-1})]$$

hence, by the concavity of the logarithm function:

$$\log(\mathbb{E}(Z^V(\underline{i}_k))) \geq \sum_{\substack{U \in \{A,B,C\} \\ a_U \neq 0}} \frac{a_U}{\gamma_j} \log(\gamma_j \mathbb{E}(Z^U(\underline{i}_{k-1})))$$

$$= \frac{a_A + a_B + a_C}{\gamma_j} \log(\gamma_j) + \sum_{\substack{U \in \{A,B,C\} \\ a_U \neq 0}} \frac{a_U}{\gamma_j} \log(\mathbb{E}(Z^U(\underline{i}_{k-1}))).$$

First we calculate the expectation of the first part:

$$\mathcal{E}\left[\frac{a_A + a_B + a_C}{\gamma_j} \log(\gamma_j)\right] = \mathcal{E}[\log(\gamma_j)] = \frac{1}{3}\log(\Gamma), \tag{3.6}$$

and now we lower bound the more complicated second part:

$$\mathcal{E}\left[\sum_{\substack{U\in\{A,B,C\}\\ a_U\neq 0}} \frac{a_U}{\gamma_j}\log(\mathbb{E}(Z^U(\underline{i}_{k-1})))\right]$$

$$=\sum_{\substack{U\in\{A,B,C\}\\ a_U\neq 0}} \mathcal{E}\left[\frac{a_U}{\gamma_j}\right]\mathcal{E}[\log(\mathbb{E}(Z^U(\underline{i}_{k-1})))]$$

$$\geq \sum_{\substack{U\in\{A,B,C\}\\ a_U\neq 0}} \mathcal{E}\left[\frac{a_U}{\gamma_j}\right](k-1)\log(\sqrt[3]{\Gamma})$$

where the first equality follows from the fact, that the choice of the next column is independent of the choice of the earlier columns, and the second inequality follows from the induction hypothesis.

$$\sum_{\substack{U\in\{A,B,C\}\\ a_U\neq 0}} \mathcal{E}\left[\frac{a_U}{\gamma_j}\right](k-1)\log(\sqrt[3]{\Gamma}) = (k-1)\log(\sqrt[3]{\Gamma})\mathcal{E}\left[\frac{1}{\gamma_j}\sum_{\substack{U\in\{A,B,C\}\\ a_U\neq 0}} a_U\right]$$

$$= (k-1)\log(\sqrt[3]{\Gamma}). \quad (3.7)$$

Consequently adding up (3.6) and (3.7) leads to:

$$\mathcal{E}[\log(\mathbb{E}(Z^V(\underline{i}_k)))] \geq \log(\sqrt[3]{\Gamma}) + (k-1)\log(\sqrt[3]{\Gamma}) = k\log(\sqrt[3]{\Gamma}).$$

Hence we verified (3.5). The next step is to show that (3.4) holds. Observe that $\mathcal{Z}_1(\bar{\theta})|\theta_0 = (i_1,\ldots,i_N)$ is the number of level N triplets coming from the first level 1 triplet in the column $\mathcal{C}_{1,1,i_1,\ldots,i_N}$. This can be lower bounded with the number of level N triplets coming from the level $N-1$ $U(i_N)$ shapes in the column $\mathcal{C}_{1,1,i_1,\ldots,i_N}$. Therefore

$$\mathbb{E}(\mathcal{Z}_1(\bar{\theta})|\theta_0 = (i_1,\ldots,i_N)) \geq \mathbb{E}(Z^{U(i_N)}(i_1,\ldots,i_{N-1}))\cdot q(i_N, U(i_N))$$

$$\geq \mathbb{E}(Z^{U(i_N)})\cdot q$$

hence by the previous inequality and (3.5):

$$\mathcal{E}(\log(\mathbb{E}(\mathcal{Z}_1(\bar{\theta})|\theta_0 = (i_1, \ldots, i_N))))$$
$$\geq \mathcal{E}(\log(\mathbb{E}(Z^{U(i_N)}))) + \log(q) \geq (N-1)\log(\sqrt[3]{\Gamma}) + \log(q).$$

We know that $\Gamma > 1$, hence $\log(\sqrt[3]{\Gamma}) > 0$ and $q > 0$, hence we can choose N such that

$$(N-1)\log(\sqrt[3]{\Gamma}) + \log(q) > 0. \tag{3.8}$$

In this way (3.4) is satisfied, hence from Theorem 3.1 it follows that with positive $\mathbb{P}$ probability the process does not die out with Prob probability 1. □

3.3 The case of the random Menger sponge

It follows from the definitions (see (2.7)) of the matrices $M(i)$, $i = 0, 1, 2$ that for the Menger sponge, Condition 4 holds whenever $p > 0$, because the third row of the matrices are positive in that case. We have seen at the beginning of Section 2.2 that $\gamma_0 = 8p$ and $\gamma_1 = \gamma_2 = 6p$, hence $\gamma_0 \cdot \gamma_1 \cdot \gamma_2 = 8p \cdot 6p \cdot 6p$. Thus Condition 4 is satisfied if $p > (8 \cdot 6 \cdot 6)^{-\frac{1}{3}}$. It follows that the parameter interval can be subdivided into four subintervals, as in Figure 3.2, the first and the last was already introduced in the previous chapter. The two middle intervals have the following properties:

- $\left(\frac{3}{20}, (8 \cdot 6 \cdot 6)^{-\frac{1}{3}}\right)$, in this thesis we don't cover this interval, but it is worth mentioning, that in an article which is under preparation we verified that for some $\hat{p} > \frac{3}{20}$ if $\frac{3}{20} < p < \hat{p}$, then $proj(\mathcal{M}_p)$ has 0

The projection has zero Lebesgue measure a.s.

The projection has positive Lebesgue measure but does not contain an interval a.s. conditioned on non-extinction

The projection contains an interval a.s. conditioned on non-extinction

D   L   C
0.148   0.15   0.152   0.154   0.156   0.158   0.16   0.162   0.164   0.166   0.168   0.17

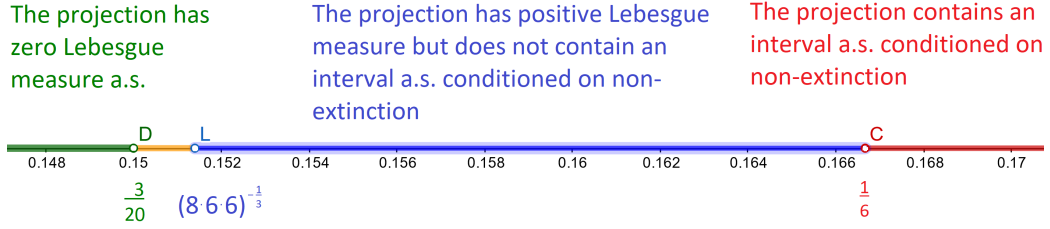$\frac{3}{20}$ $(8 \cdot 6 \cdot 6)^{-\frac{1}{3}}$ $\frac{1}{6}$

Figure 3.2: The parameter intervals for p in the case of the random Menger sponge.

Lebesgue-measure almost surely despite the fact that $\mathcal{M}_p$ has Hausdorff dimension greater than 1 almost surely conditioned on non extinction.

- $\left((8 \cdot 6 \cdot 6)^{-\frac{1}{3}}, \frac{1}{6}\right)$, choosing p from this interval give rise to having an almost sure positive measure projection conditioned on non-extinction, however with probability one the projection does not contain an interval, as it was shown in the previous chapter.

Chapter 4

Conclusions

In this Thesis we had been proven two theorems about the projection of the three-dimensional inhomogeneous Mandelbrot percolation fractal, although the most interesting part is the example – the Menger sponge. In [9] Károly Simon and Lajos Vágó proved (among other things) that in three dimension (and in two also, see [7]) in the homogeneous case whenever $\dim_H \Lambda > 1$ the projection contains an interval almost surely conditioned in non extinction, and when $\dim_H \Lambda \leq 1$ the projection has zero Lebesgue measure almost surely. We had seen in two dimensions ([8]) that this is not the case when we have inhomogeneous probabilities, that is the two authors had shown that for the random Sierpiński carpet, it is possible that $dim_H(\Lambda) > 1$ (a.s. conditioned on non extinction) and concurrently the projection to one of the coordinate axes does not contain an interval. What we know of the Menger sponge is more, we mentioned – although not proved – that it is possible that $dim_H(\mathcal{M}_p) > 1$ and the Lebesgue measure of the projection is zero, we also proved that it is possible that the Lebesgue measure of the projection is
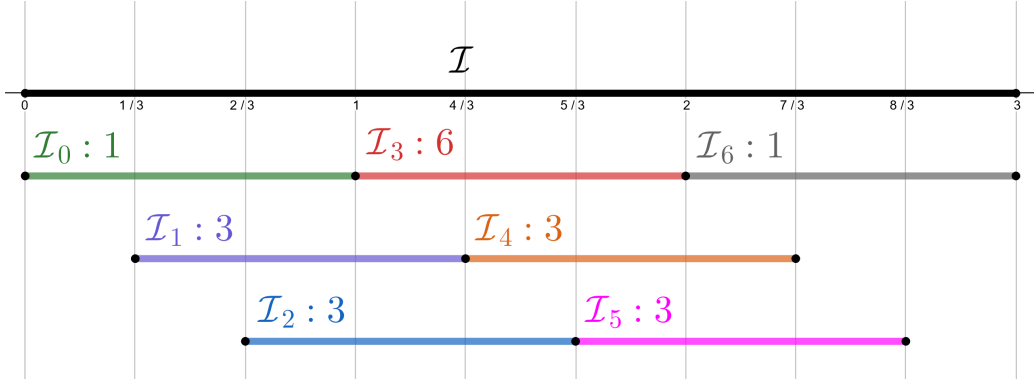
Figure 4.1: The projection of the Menger sponge, level 1. The notation is Name of the interval: number of cubes which projection is the given interval.

not zero, but it does not contain an interval. This second phenomenon is of further interest as we explain.

It is an open question whether or not there exists a (deterministic) self-similar set on the line with positive Lebesgue measure and empty interior, for more details see [2]. We now show an example of such a set, although not in the deterministic but the random case. Instead of the random Menger sponge we turn our attention to the projected IFS (see Figure 4.1), namely:

$$\widetilde{proj} : \mathbb{R}^3 \to \mathbb{R}, \quad \widetilde{proj}(x, y, z) = x + y + z \tag{4.1}$$

and

$$\tilde{S}_{i,j,k} = \widetilde{proj} \circ S_{i,j,k} \circ \widetilde{proj}^{-1}, \text{ for } i, j, k \in \{0, 1, 2\} \tag{4.2}$$

$S_{i,j,k}$ was defined in (3.4). Now consider the random attractor (denote it $\widetilde{\Lambda}_p$) of the IFS $\tilde{\mathcal{S}} = \{\widetilde{proj} \circ S_{i,j,k}\}_{(i,j,k) \in \mathcal{J}}$ (for the definition of $\mathcal{J}$ see (see (1.4)), which we get by applying the rules of the homogeneous Mandelbrot percolation with a parameter $p \in [0, 1]$. $\widetilde{\Lambda}_p$ has the same distribution as

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$proj(\mathcal{M}_p)$, hence using the results of this thesis for $p \in \left((8 \cdot 6 \cdot 6)^{-\frac{1}{3}}, \frac{1}{6}\right)$: $\widetilde{\Lambda}_p$ has empty interior almost surely, and positive Lebesgue measure almost surely conditioned on non-extinction.

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