

PROJECTIONS OF THE RANDOM MENGER SPONGE

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ABSTRACT. In this paper we prove theorems about a special family of random self-similar sets on the line and we apply these theorems to get the Hausdorff dimension, the Lebesgue measure and existence of interior points of some projections of the random right angled Sierpiński gasket, the random Sierpiński carpet and the random Menger sponge. The Menger sponge is one of the most well-known example of self-similar sets in \mathbb{R}^3 . The Mandelbrot percolation process restricted to the cubes, which are the building blocks of the Menger sponge, yields the random Menger sponge, a random self-similar fractal in \mathbb{R}^3 . We examine its orthogonal projections to straight lines, from the point of Lebesgue measure and existence of interior points. In particular this yields random self-similar sets on the line with positive Lebesgue measure and empty interior. Moreover, we give a sharp threshold for the probability above which the projections of the random Menger sponge contains an interval in all directions.

Dedicated to the memory of Professor Ka-Sing Lau.

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1. INTRODUCTION

1.1. Brief summary. The Mandelbrot percolation Cantor set is a two-parameter family of random sets on \mathbb{R}^d . Namely, fix the parameters $K \geq 2$ and $p \in (0, 1)$. In the first step of the construction we partition the unit cube $[0, 1]^d$ into axes-parallel cubes of side length $1/K$. Each of these cubes are retained with probability p and discarded with probability $1 - p$ independently. This step is repeated independently in each of the retained cubes *ad infinitum* or until no retained cubes are left. The resulting random set is the Mandelbrot percolation set.

Falconer and Jin introduced a generalization of the Mandelbrot percolation Cantor sets [10]. In this paper we consider a special case of Falconer and Jin's construction.

Namely, we consider a (deterministic) M -ary tree \mathcal{T} . That is, every node of \mathcal{T} has exactly M children. (In the construction of the Mandelbrot percolation set $M = K^d$.) We assign a random label (from $\{0, 1\}$) to each of these nodes. The label of the root \emptyset is equal to 1 and the random label of all other nodes are independent Bernoulli(p) random variables. A level n node is retained if all of its ancestors are labelled with 1. In the Mandelbrot percolation example, every retained level- n node naturally corresponds to a retained level n cube. An infinite path starting from the root is retained if all the nodes of the path are labelled with 1. It may happen that no infinite paths are retained. This event is called extinction. The set of retained level n nodes is denoted by \mathcal{E}_n for an $n \in \mathbb{N} \cup \{\infty\}$. In the case of the Mandelbrot percolation, every element of \mathcal{E}_∞ naturally correspond to a point of the Mandelbrot percolation Cantor set.

A generalization of the Mandelbrot percolation sets can be obtained if we consider a self-similar IFS (Iterated Function System) $\mathcal{F} := \{f_i\}_{i=1}^M$ on \mathbb{R}^d and, retain the points of the attractor of \mathcal{F} having a symbolic representation in \mathcal{E}_∞ . We call these random sets **coin tossing self-similar sets** since we decide if a cylinder set is retained or not as a result of subsequent coin-tossings. For more detailed description see Definition 1.1.

1.2. Notations. Before we give the precise definition of the coin tossing self-similar sets, first we define deterministic self-similar sets in \mathbb{R}^d . Fix a self-similar IFS \mathcal{F} on \mathbb{R}^d ,

$$(1.1) \quad \mathcal{F} := \{f_i(x) := r_i Q_i x + t_i\}_{i=0}^{M-1}, \quad f_i : \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad r_i \in (0, 1), \quad Q_i \in O(d), \quad t_i \in \mathbb{R}^d.$$

We use the short-hand notations

$$f_{i_1, \dots, i_n} := f_{i_1} \circ \dots \circ f_{i_n}, \quad r_{i_1, \dots, i_n} := r_{i_1} \dots r_{i_n}, \quad [M] := \{0, \dots, M-1\}.$$

It is easy to see that we can choose

$$(1.2) \quad B \subset \mathbb{R}^d \text{ compact such that } f_i(B) \subset B \text{ for all } i \in [M].$$

Then the union of all n -cylinders $\bigcup_{i_1, \dots, i_n} f_{i_1, \dots, i_n}(B)$ form a nested sequence of compact sets. Their intersection is the attractor

$$\Lambda := \bigcap_{n=1}^{\infty} \bigcup_{i_1, \dots, i_n} f_{i_1, \dots, i_n}(B).$$

The definition of Λ does not depend on the choice of B as long as B satisfies (1.2).

Definition 1.1 (Coin tossing self-similar sets). *Let $\mathcal{F} := \{f_i\}_{i=0}^{M-1}$ be a (deterministic) self-similar IFS on \mathbb{R}^d as it was defined in (1.1) and let $p \in (0, 1)$. The corresponding coin tossing self-similar set $\Lambda_{\mathcal{F}}(p)$ is defined as follows: In the first step for every $k \in [M]$ we toss (independently) a biased coin which lands on head with probability p . The random subset $X_1 \subset [M]$ consists of those $k \in [M]$ for which the coin tossing resulted in head. Assume that we have already constructed $X_n \subset [M]^n$. Then for every node $\mathbf{i} \in X_n$ we define (independently of everything) the random set $X_1^{\mathbf{i}} \subset [M]$ which has the same distribution as X_1 . The set of the offspring of \mathbf{i} is defined by $O(\mathbf{i}) = \{\mathbf{ik} \in [M]^{n+1} : k \in X_1^{\mathbf{i}}\}$, where $\mathbf{ik} = \mathbf{i}_1, \dots, \mathbf{i}_n, k$ if $\mathbf{i} = \mathbf{i}_1, \dots, \mathbf{i}_n$. Finally, we form $X_{n+1} = \bigcup_{\mathbf{i} \in X_n} O(\mathbf{i}) \subset [M]^{n+1}$. Then the coin tossing self-similar set is defined by*

$$\Lambda_{\mathcal{F}}(p) := \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{i} \in X_n} f_{\mathbf{i}}(B),$$

where B is chosen as in (1.2).

We do not assume that the Open Set Condition (OSC) (see [12]) holds for \mathcal{F} . However, we mention the following theorem.

Theorem 1.2 (Falconer [7], Mauldin-Williams [14]). *Let \mathcal{F} be a deterministic self-similar IFS, as in (1.1), which satisfies the OSC. Then for the coin tossing self-similar set $\Lambda_{\mathcal{F}}(p)$, we have*

$$(1.3) \quad \dim_{\text{H}} \Lambda_{\mathcal{F}}(p) = \dim_{\text{B}} \Lambda_{\mathcal{F}}(p) = s, \text{ where } \sum_{i=0}^{M-1} pr_i^s = 1,$$

almost surely, conditioned on non-extinction, where r_i is the contraction ratio of the similarity mapping f_i .

Motivated by this formula we introduce the similarity dimension of a coin tossing self-similar set $\Lambda_{\mathcal{F}}(p)$:

$$\dim_{\text{Sim}} \Lambda_{\mathcal{F}}(p) := s, \text{ where } \sum_{i=0}^{M-1} pr_i^s = 1.$$

In this paper we only consider homogeneous coin tossing self-similar IFSs, which means that all contraction ratios r_i are equal to the same $r \in (0, 1)$. In this homogeneous case, formula (1.3) simplifies to

$$(1.4) \quad (\text{OSC} \ \& \ r_i \equiv r, \ \forall i) \implies s = \dim_{\text{H}} \Lambda_{\mathcal{F}}(p) = \dim_{\text{B}} \Lambda_{\mathcal{F}}(p) = \frac{\log(Mp)}{-\log r},$$

almost surely, conditioned on non-extinction.

A more detailed definition of a coin tossing self-similar set, which describes the ambient probability space can be found in Section 3.1.

Example 1.3 ((Homogeneous) Mandelbrot percolation). *The homogeneous Mandelbrot percolation on \mathbb{R}^d with parameters (K, p) can be obtained as a special case of the construction defined above by choosing $M = K^d$ and*

$$\mathcal{F} := \left\{ f_i(x) = \frac{1}{K}x + t_i \right\}_{i=0}^{M-1},$$

where $\{t_i\}_{i=0}^{M-1}$ is an enumeration of the left bottom corners of the K -mesh cubes contained in $[0, 1]^d$.

If we project a d -dimensional coin tossing self-similar set to straight lines, then the resulting random sets are coin tossing self-similar sets on the line.

Example 1.4 (Coin tossing integer self-similar sets on the line). *We obtain the coin tossing integer self-similar sets on the line (with parameters \mathcal{F} and p) by applying the random construction introduced in Definition 1.1 for the following deterministic IFS:*

$$\mathcal{F} := \left\{ f_i(x) := \frac{1}{L}x + t_i \right\}_{i=0}^{M-1}, \quad f_i : \mathbb{R} \rightarrow \mathbb{R}, \quad L \in \mathbb{N} \setminus \{0, 1\}, \quad t_i \in \mathcal{N},$$

where $\mathcal{N} \subset \mathbb{R}$ is a lattice.

Remark 1.5. *Without loss of generality (see [2, Section 1.3.3]) we may assume, that*

$$\mathcal{N} := \mathbb{N}, \quad 0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_{M-1}, \quad \text{and} \quad L - 1 \mid t_{M-1}.$$

Remark 1.6. *In the deterministic case we usually require that the elements of the IFS \mathcal{F} are different. However, in the random case we allow repetition among the functions of \mathcal{F} . This is reasonable since even if $f_i = f_j$ we randomize them differently. For example, if $\mathcal{F} := \{\frac{1}{3}x, \frac{1}{3}x, \frac{1}{3}x + \frac{2}{3}\}$ and $\mathcal{S} := \{\frac{1}{3}x, \frac{1}{3}x + \frac{2}{3}\}$, then the coin tossing self-similar sets $\Lambda_{\mathcal{F}}(p)$, $\Lambda_{\mathcal{S}}(p)$ are different.*

Note that the distribution of the two Mandelbrot percolation is different even if instead of $\Lambda_{\mathcal{S}}(p)$ we consider the system, where the assigned probabilities are $\min\{1, 2p\}$ and p respectively. If $p \leq \frac{1}{2}$ in this case the probability that in the first step we discard $[0, \frac{1}{3}]$ is $1 - 2p$, whereas in the case of $\Lambda_{\mathcal{F}}(p)$ this probability is $(1 - p)^2$.

Definition 1.7. *Given the deterministic IFS $\mathcal{F} := \{f_i\}_{i=0}^{M-1}$, we denote by $\mathcal{S} := \{S_i\}_{i=0}^{m-1} \subset \mathcal{F}$ the IFS consisting of the distinct elements of \mathcal{F} . That is for every $i \in [M]$ there exist a unique $j \in [m]$ such that $f_i = S_j$. For every $j \in [m]$ let*

$$(1.5) \quad n_j := \#\{f_i \in \mathcal{F} : f_i = S_j\}.$$

For $j \in \{0, \dots, m-1\}$, we define q_j, ℓ_j and k_j in the following way,

$$(1.6) \quad q_j := \frac{n_j}{M} = \frac{k_j}{\ell_j}, \quad k_j, \ell_j \in \mathbb{Z} \setminus \{0\}, \quad \gcd(k_j, \ell_j) = 1.$$

The corresponding probability vector is

$$\mathbf{q} := (q_1, \dots, q_m).$$

For $k \geq 2$ we write $\Sigma^{(k)} := [k]^{\mathbb{N}}$, recalling, that $[k] := \{0, 1, \dots, k-1\}$.

We introduce the natural (probability) measure $\mu := \mathbf{q}^{\mathbb{N}}$ on $\Sigma^{(m)}$. We define the natural projection $\Pi^{(m)} : \Sigma^{(m)} \rightarrow \mathbb{R}$

$$\Pi^{(m)}(\mathbf{i}) := \lim_{n \rightarrow \infty} S_{i_1, \dots, i_n}(0),$$

for $\mathbf{i} = i_1, \dots, i_n, \dots \in \Sigma^{(m)}$. The push forward of the natural measure μ is denoted by ν ,

$$(1.7) \quad \nu := \Pi_*^{(m)} \mu.$$

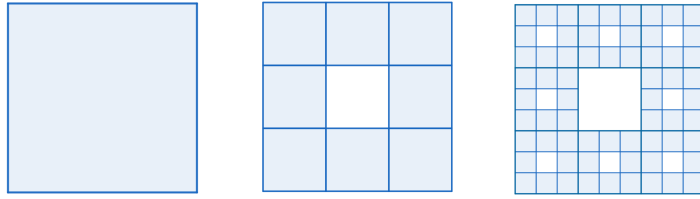


FIGURE 1. The first three level approximation of the Sierpiński carpet.

Following Ruiz [17, Section 3.1.1] we define the L-adic intervals:

$$\mathcal{D}_k := \{ [(i-1) \cdot L^{-k}, i \cdot L^{-k}] : i \in \mathbb{Z} \}, \quad \text{for } k \in \{-1, 0, \dots\}.$$

Since ν is compactly supported there exist finitely many intervals, called basic types,

$$(1.8) \quad J^0, \dots, J^{N-1} \in \mathcal{D}_{-1} \text{ such that}$$

$$(1.9) \quad \text{spt}(\nu) \subset \cup_{i \in [N]} J^i \text{ and } \nu(J^i) > 0 \text{ for every } i \in [N],$$

where we assume that the intervals J^0, \dots, J^{N-1} are arranged in increasing order. It is clear from the definition that the interval spanned by the attractor is $I := [0, L \frac{t_{m-1}}{L-1}]$. It follows from our assumption $L-1 | t_{m-1}$ that the right endpoint of J^{N-1} coincides with the right endpoint of I . For every $k \in [N]$ the interval J^k subdivides into L^n intervals from \mathcal{D}_{n-1} (of length $L^{-(n-1)}$) which are denoted by $J_{\mathbf{a}}^k = J_{a_1, \dots, a_n}^k$ for $\mathbf{a} = (a_1, \dots, a_n) \in [L]^n$. We define the $N \times N$ matrices $\{A_{\mathbf{a}}\}_{\mathbf{a}=0}^{L-1}$:

$$(1.10) \quad A_{\mathbf{a}}(\ell, k) := \begin{cases} n_i, & \text{if } \exists i \quad S_i(J^k) = J_{\mathbf{a}}^\ell; \\ 0, & \text{otherwise.} \end{cases}$$

For the usage of these definitions in a concrete, simple and important example see Example 4 about the orthogonal projections of the Sierpiński carpet below.

Note that we index the rows and columns of the matrices $A_{\mathbf{a}}$ from 0 to $N-1$. For $\mathbf{a} = (a_1, \dots, a_n) \in [L]^n$,

$$A_{\mathbf{a}} := A_{a_1} \cdots A_{a_n}.$$

The meaning of $A_{\mathbf{a}}(\ell, k)$ is simply the number of indices $(i_1, \dots, i_n) \in [M]^n$ such that $f_{i_1, \dots, i_n}(J^k) = J_{\mathbf{a}}^\ell$ (this is stated later, in Lemma 3.3). The j -th column sum (CS) of the \mathbf{a} -th matrix is denoted by

$$CS_{\mathbf{a}, j} := \sum_{i=0}^{N-1} A_{\mathbf{a}}(i, j).$$

Example 1.8 (Random Sierpiński carpet). *The (deterministic) Sierpiński carpet is the attractor (see Figure 1) of the following self-similar IFS in \mathbb{R}^2 :*

$$\mathcal{F} := \left\{ f_i(\mathbf{x}) = \frac{1}{3}(\mathbf{x}) + t_i \right\}_{i=0}^8,$$

where $\{t_i\}_{i=0}^8$ is an enumeration of the set

$$\left\{ 0, \frac{1}{3}, \frac{2}{3} \right\}^2 \setminus \left\{ \left(\frac{1}{3}, \frac{1}{3} \right) \right\}.$$

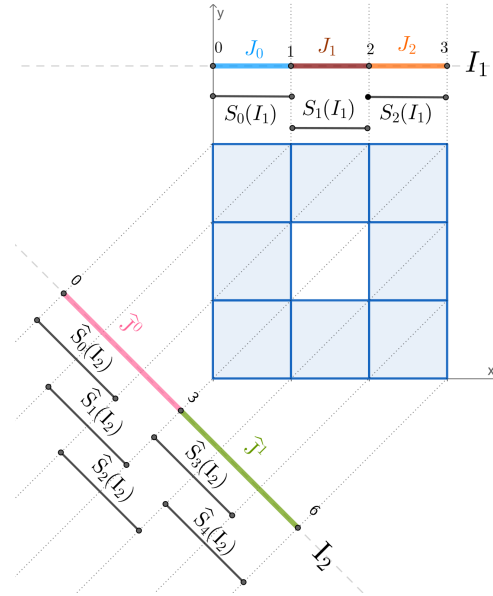


FIGURE 2. Illustration of the $\widehat{\text{proj}}_{(1,-1)}$ and the $\widehat{\text{proj}}_{(1,0)}$ projection of the first level of the Sierpiński carpet.

We obtain the random Sierpiński carpet by applying the random construction introduced in Definition 1.1 for the deterministic IFS above. We denote the random Sierpiński carpet with parameter p by \mathcal{S}_p .

Let $\widehat{\text{proj}}_{(\alpha,\beta)}$ ($\alpha, \beta \in \mathbb{R}$) denote the following projection to \mathbb{R} :

$$\widehat{\text{proj}}_{(\alpha,\beta)}(x, y) = \alpha \cdot x + \beta \cdot y,$$

which is the orthogonal projection to the line of tangent β/α rescaled. In what follows we consider the $(1, -1)$ and the $(1, 0)$ projections of the random Sierpiński carpet as it is schematically illustrated in Figure 2.

First consider $\widehat{\text{proj}}_{(1,0)}(\mathcal{S}_p)$. The projected and rescaled IFS is the following (see Definition 1.7): $S_i(x) = \frac{1}{3}x + t_i$, $t_i \in \{0, 1, 2\}$, with $(n_0, n_1, n_2) = (3, 2, 3)$, and basic type (see (1.8)) $J^0 = [0, 3]$. Hence, the corresponding 1×1 matrices (see (1.10)) are,

$$D_0 = [3], \quad D_1 = [2], \quad D_2 = [3].$$

In this case the spectral radiuses of $p \cdot D_i$, $i \in \{0, 1, 2\}$ are $3p, 2p, 3p$ respectively.

Secondly, consider $\widehat{\text{proj}}_{(1,-1)}(\mathcal{S}_p)$. The projected and rescaled IFS is the following: $\widehat{S}_i(x) = \frac{1}{3}x + t_i$, $t_i \in \{0, 1, 2, 3, 4\}$, with $(n_0, n_1, n_2, n_3, n_4) = (1, 2, 2, 2, 1)$, and basic types $\widehat{J}^0 = [0, 3]$, $\widehat{J}^1 = [3, 6]$. Hence, the corresponding matrices are,

$$C_0 = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}.$$

The spectral radiuses of $p \cdot C_i$ ($i \in \{0, 1, 2\}$) are $2p, 3p, 2p$ respectively.

In what follows we present our results regarding general self-similar coin tossing IFSs on the line. After, we use these results to investigate some orthogonal projections of random self-similar carpets; the random Menger sponge the random Sierpiński carpet and the random right-angled Sierpiński gasket. In this section we also state a theorem about all of the projections of the random Menger-sponge.

1.3. Results for general self-similar coin tossing IFSs on the line. In the rest of this section we deal with coin tossing integer self-similar IFSs \mathcal{F} on the line introduced in Example 1.4 including Remark 1.5. That is for the rest of this section we always assume that

$$(1.11) \quad \mathcal{F} := \left\{ f_i(x) := \frac{1}{L}x + t_i \right\}_{i=0}^{M-1},$$

$$L \in \mathbb{N} \setminus \{0, 1\}, t_i \in \mathbb{N}, 0 = t_0 \leq \dots \leq t_{M-1}, L - 1 \mid t_{M-1}.$$

Moreover, throughout this Section we use the notation introduced in Section 1.2. The new results of this paper are as follows:

Theorem 1.9. *The coin tossing integer self-similar set $\Lambda_{\mathcal{F}}(p)$ CONTAINS AN INTERVAL, almost surely conditioned on non-extinction, if the following two conditions hold*

- (1) $p \cdot CS_{a,U} > 1$ for all $a \in [L]$ and $U \in [N]$. That is, p is larger than the reciprocal of every column sum of every matrix.
- (2) There exist $\mathbf{b} \in [L]^*$ and $U \in [N]$ such that $A_{\mathbf{b}}(U, V) > 0$ for all $V \in [N]$. That is there exist a product $(A_{\mathbf{b}}, \mathbf{b} \in [L]^*)$ of the matrices with a strictly positive row.

Theorem 1.10. *The coin tossing integer self-similar set $\Lambda_{\mathcal{F}}(p)$ DOES NOT CONTAIN ANY INTERVALS, almost surely, if there exists an $a \in [L]$ such that the spectral radius of the matrix $p \cdot A_a$ is strictly smaller than 1.*

Theorem 1.11. *The coin tossing integer self-similar set $\Lambda_{\mathcal{F}}(p)$ HAS POSITIVE LEBESGUE MEASURE, almost surely conditioned on non-extinction, if the following two conditions hold:*

- (1) $p^L \cdot \left(\prod_{a=0}^{L-1} CS_{a,U} \right) > 1$ for all $U \in [N]$. That is for every column index $U \in [N]$ we consider the geometric mean of the U -th column sums of the matrices $\{A_a\}_{a=0}^{L-1}$ and denote it by g_U . Our assumption is that $p > \max_{U \in [N]} \frac{1}{g_U}$.
- (2) For every $b \in [L]$ there exists an $U \in [N]$ such that $A_b(U, V) > 0$ for all $V \in [N]$. That is, every matrix A_b ($b \in [L]$) has a row with all positive elements.

Theorem 1.12. *Assume that for $R := \text{lcm}(\ell_0, \dots, \ell_{m-1})$ (recall that ℓ_j was defined in (1.6)) $L \nmid R$ and the deterministic attractor $\Lambda_{\mathcal{F}}$ has positive Lebesgue measure. Then there exists a $p_0 > \frac{L}{M}$ such that for all $p \in (0, p_0)$ the UPPER BOX DIMENSION of the coin tossing integer self-similar set $\Lambda_{\mathcal{F}}(p)$ is almost surely SMALLER THAN ONE,*

$$\overline{\dim_B}(\Lambda_{\mathcal{F}}(p)) < 1.$$

The relevance of the bound $\frac{L}{M}$ is that the similarity dimension $\dim_{\text{Sim}} \Lambda_{\mathcal{F}}(p) > 1$ if and only if $p > \frac{L}{M}$. Thus, for $p \in \left(\frac{L}{M}, B\right)$ the similarity dimension of $\Lambda_{\mathcal{F}}(p)$ is greater than 1 but its upper box-dimension is smaller than one.

1.3.1. The organization of the rest of the paper. In Section 2 we show some applications of Theorem 1.9-1.12 on orthogonal projections of the randomized version of some well-known fractal sets. In particular first we consider the orthogonal projection of the random-Menger sponge (for the definition see section 2.1) to the space diagonal of the unit cube. We also state a theorem which states that if the parameter p is greater than 0.25 then all orthogonal projections of the random Menger sponge contains an interval almost surely and that this is a sharp bound, since the projections to the coordinate axes has empty interior almost surely whenever $p \leq 0.25$. This theorem is proven in section 4. Secondly, we consider the orthogonal projections of the random Sierpiński carpet. The behaviour of the projection of the random Sierpiński carpet to the coordinate axes is well-known

(see [8] and [5]). In this paper we consider the 45-degree projection as well, which does not satisfy the OSC. Our last example is the 45-degree projection of the right angled Sierpiński gasket.

In section 3 we present the proof of the general theorems, Theorem 1.9-1.12. We start by proving Theorem 1.9 using a similar branching process type argument as in [6], secondly we prove Theorem 1.10 by a standard argument, then we prove Theorem 1.11 in a similar way as in [15]. Finally, for the proof of Theorem 1.12, we use the results of Bárány and Rams [3], although our setup is more general (as in [17]). In the Appendix we prove that the results of [3] holds for not just projections of two-dimensional carpets but in our—more general—setup as well. Before that in section Random Menger sponge (section 4) using the result of Vágó Lajos and the second author ([18]) we prove the above mentioned theorem about the interior points of all the projections of the random Menger sponge.

Remark 1.13. *In a forthcoming paper we prove the following related assertion: Consider the matrices $\{A_a\}_{a=1}^L$ that were defined in (1.10). Let $\lambda(\mathcal{F}, p)$ be the Lyapunov exponent of the corresponding random matrix product. That is $\lambda(\mathcal{F}, p) := \lim_{n \rightarrow \infty} \frac{1}{n} \log(p^n \|A_{a_1 \dots a_n}\|)$ for a \tilde{L} -typical $\mathbf{a} = (a_1, a_2, \dots) \in [L]^{\mathbb{N}}$, where \tilde{L} is the uniform distribution measure on $[L]^{\mathbb{N}}$. Then under some mild conditions we have*

- (a) *if $\lambda(\mathcal{F}, p) > 0$ then $\Lambda_{\mathcal{F}}(p)$ has positive Lebesgue measure (conditioned on non-extinction).*
- (b) *If $\lambda(\mathcal{F}, p) < 0$ then $\Lambda_{\mathcal{F}}(p)$ has zero Lebesgue measure.*

This sharper theorem mentioned above does not make Theorem 1.11 redundant since Theorem 1.11 gives checkable conditions while we do not have any efficient methods for the numerical computation of $\lambda(\mathcal{F}, p)$.

1.3.2. *Open problems.* There are some problems that are not covered in this paper even though are worth mentioning. One of these open questions is the formula for the Hausdorff dimension of $\Lambda_{\mathcal{F}}(p)$. Secondly we conjecture though not proved that the box and the Hausdorff dimensions of $\Lambda_{\mathcal{F}}(p)$ coincide. And lastly we conjecture that in those cases when the reduced (\mathcal{S} , see Definition 1.7) IFS have overlaps there exists $p_1 < p_2$ such that for $p \in (p_1, p_2)$ the following holds $\dim_{\text{H}} \Lambda_{\mathcal{F}}(p) = 1$ but $\mathcal{L}eb_1(\Lambda_{\mathcal{F}}(p)) = 0$.

2. THE PROJECTIONS OF THE RANDOM MENGER SPONGE, THE RANDOM SIERPIŃSKI CARPET AND THE RANDOM RIGHT-ANGLED SIERPIŃSKI GASKET

In this section our goal is to show some interesting application of the general theorems (Theorem 1.9-1.12) to special projections of random carpets. The projections (especially to the coordinate axes) of the random Sierpiński carpet are very well studied (see [16], [19], [5]). Here we summarize some of the results that are already known or can be easily deduced from the literature and we extend these in the case of the projection to the diagonal of the unit square. In this case the projected IFS contains overlaps, which makes the situation more complicated. A 3-dimensional analogue of the Sierpiński carpet is the Menger sponge. The projection of the right-angled Sierpiński gasket to the diagonal of the unit square is a special example where the IFS does not contain any duplicates.

2.1. The random Menger sponge. The (deterministic) Menger sponge is the attractor (see Figure 3) of the following self-similar IFS in \mathbb{R}^3 :

$$\mathcal{F} := \left\{ f_i(\mathbf{x}) = \frac{1}{3}(\mathbf{x}) + t_i \right\}_{i=0}^{19},$$

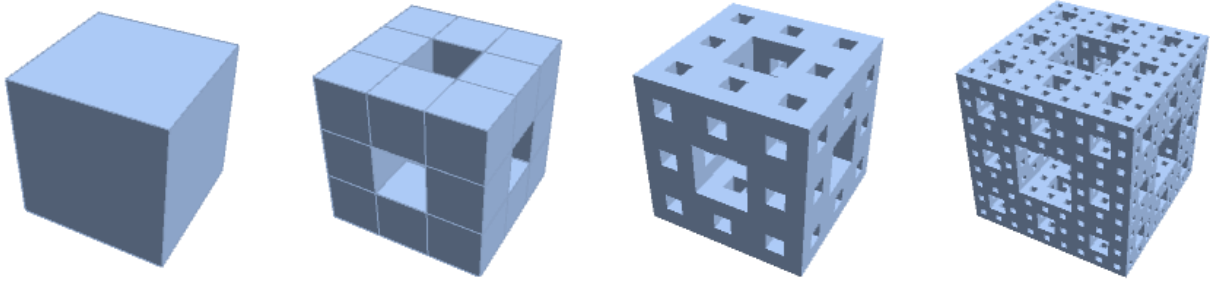


FIGURE 3. The zeroth, first, second and third level approximation of the Menger sponge.

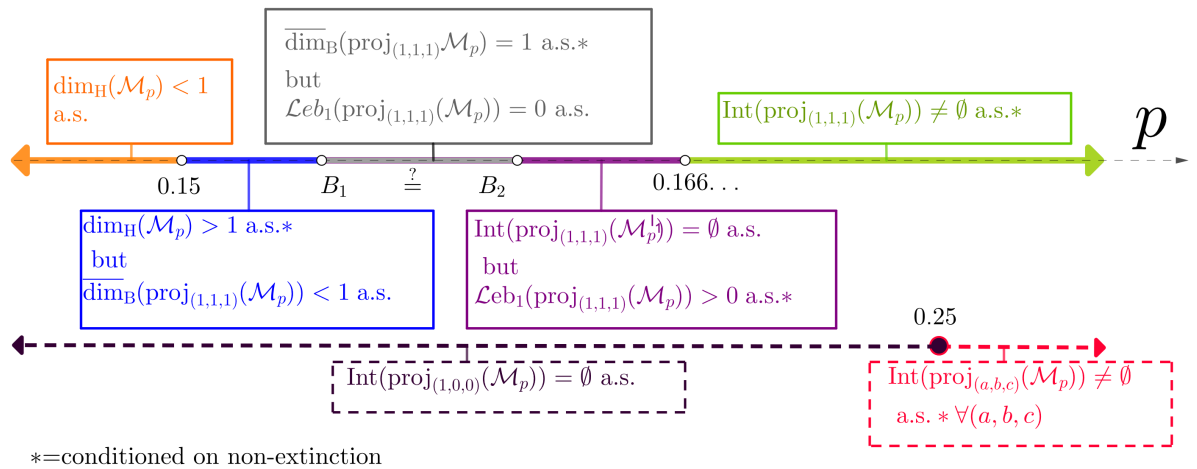


FIGURE 4. The phase transitions of the Menger sponge as described in Theorem 2.1 explained. Note that the existence of the grey($[B_1, B_2]$) interval is currently unknown.

where $\{t_i\}_{i=0}^{19}$ is an enumeration of the set

$$\left\{0, \frac{1}{3}, \frac{2}{3}\right\}^3 \setminus \left\{\left(\frac{1}{3}, \frac{1}{3}, 0\right), \left(\frac{1}{3}, 0, \frac{1}{3}\right), \left(0, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}.$$

We obtain the random Menger sponge by applying the random construction introduced in Definition 1.1 for the deterministic IFS above. We denote the random Menger sponge with parameter p by \mathcal{M}_p . Let $\text{proj}_{(a,b,c)}(x, y, z) : \mathbb{R}^3 \rightarrow \mathbb{R}$ denote the scalar multiple of the orthogonal projection to the vector (a, b, c) ,

$$\text{proj}_{(a,b,c)}(x, y, z) := ax + by + cz,$$

and the projection is rational if $a, b, c \in \mathbb{Q}$.

We consider the projection $\text{proj}_{(1,1,1)}$ of \mathcal{M}_p . Using that all rational projections of \mathcal{M}_p (after proper re-scaling) can be considered as coin tossing integer self-similar sets, the following theorem is an easy consequence of our more general theorems (Theorem 1.9-1.12). It follows from (1.4) that $\dim_{\text{H}} \mathcal{M}_p = \frac{\log 20}{\log 3}$, hence $\dim_{\text{H}} \mathcal{M}_p > 1$ almost surely conditioned on non-extinction if and only if $p > \frac{3}{20}$.

Theorem 2.1. Consider the random Menger sponge $\mathcal{M}(p)$.

- For every rational projection $\text{proj}_{(a,b,c)}$ there exists a $p' = p'(a, b, c) > \frac{3}{20} = 0.15$ such that for every $0.15 < p < p'$: $\overline{\dim_B(\text{proj}_{(a,b,c)}(\mathcal{M}_p))} < 1$ almost surely but $\dim_H \mathcal{M}_p > 1$ almost surely conditioned on non-extinction.
- If $p > \frac{1}{6} = 0.166\dots$, then $\text{proj}_{(1,1,1)}(\mathcal{M}_p)$ contains an interval almost surely conditioned on non-extinction, and if $p < \frac{1}{6}$, then $\text{proj}_{(1,1,1)}(\mathcal{M}_p)$ does not contain an interval almost surely.
- If $p > (8 \cdot 6 \cdot 6)^{-\frac{1}{3}} = 0.1514\dots$, then $\mathcal{L}eb_1(\text{proj}_{(1,1,1)}(\mathcal{M}_p)) > 0$ almost surely conditioned on non-extinction.

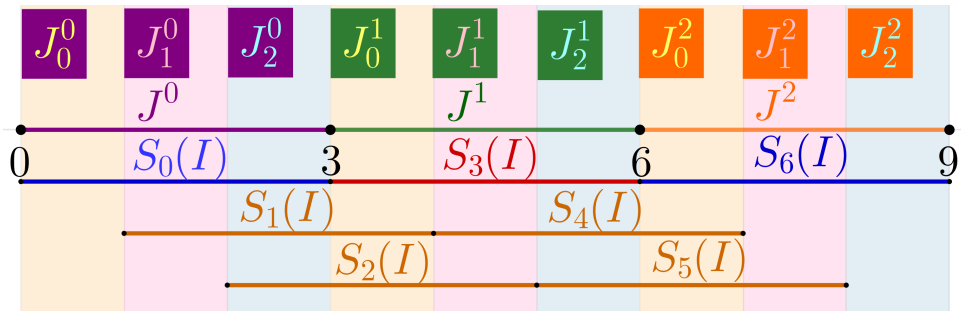


FIGURE 5. Illustration for construction of the matrices in proof of Theorem 2.1

Proof of Theorem 2.1. To prove the theorem we write the projection $\text{proj}_{(1,1,1)}(\mathcal{M}_p)$ as the attractor of the coin tossing self-similar set on the line as follows. $S_i(x) = \frac{1}{3}x + i$, $i \in [7]$. $(n_0, \dots, n_6) = (1, 3, 3, 6, 3, 3, 1)$. $J^0 = [0, 3]$, $J^1 = [3, 6]$, $J^2 = [6, 9]$. For an illustration see Figure 3.1.

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 3 & 1 & 0 \\ 3 & 6 & 3 \\ 0 & 1 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 3 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

The spectral radius of $p \cdot A_0$ is $6p$, which is less than 1 whenever $p < \frac{1}{6}$. Now if we apply Theorems 1.9-1.12 to the matrices A_0, A_1, A_2 the statement follows. \square

Theorem 2.2. Conditioned on non-extinction for almost every realization of \mathcal{M}_p for every $(a, b, c) \in \mathbb{R}^3$ direction, $\text{proj}_{(a,b,c)}(\mathcal{M}_p)$ contains an interval whenever $p > 0.25$. For $p \leq 0.25$ $\text{proj}_{(1,0,0)}(\mathcal{M}_p)$ does not contain an interval almost surely.

The second part of the theorem follows from the following description of the projected IFS and Theorem 1.10. We project the random Menger sponge to the x-axis, hence the corresponding IFS is the following (see Definition 1.7): $S_i(x) = \frac{1}{3}x + t_i$, $t_i \in \{0, 1, 2\}$, with $(n_0, n_1, n_2) = (8, 4, 8)$, and basic type (see (1.8)) $J^0 = [0, 3]$. Hence, the corresponding matrices (see (1.10)) are,

$$B_0 = [8], \quad B_1 = [4], \quad B_2 = [8].$$

The proof of the first part is presented in section random Menger sponge (see 4.1).

2.2. The random Sierpiński carpet. This example was introduced in Example 1.8, recall that we denoted the random Sierpiński carpet by \mathcal{S}_p and $\widehat{\text{proj}}_{(\alpha,\beta)}(x,y) = \alpha \cdot x + \beta \cdot y$. The behaviour of the projections of \mathcal{S}_p is quite well known (see for example [16], [19]). In particular, about the $\widehat{\text{proj}}_{(1,0)}$ projection we know almost everything from [5] (see Figure 6). Later Falconer and Grimmett [8] added to this that if $p \leq \frac{1}{2}$ then the projection $\widehat{\text{proj}}_{(1,0)}$ does not contain any intervals almost surely but if $p > \frac{1}{2}$ it contains an interval almost surely conditioned on non-extinction.

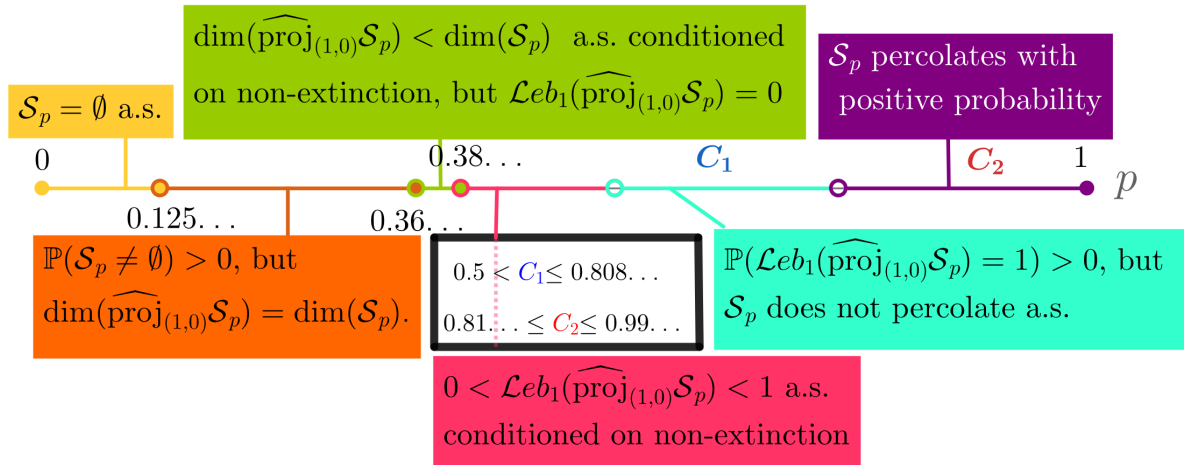


FIGURE 6. Illustration of the results of Dekking and Meester [5] on the $\widehat{\text{proj}}_{(1,0)}$ phases of the projection of the random Sierpiński carpet. It follows from the results of Falconer and Grimmett that the content of the red rectangle can be replaced with " $\widehat{\text{proj}}_{(1,0)}(\mathcal{S}_p)$ has positive Lebesgue measure but empty interior a.s. conditioned on non-extinction".

Moreover, it is stated in [16], that this last result of having non-empty interior almost surely conditioned on non-extinction, whenever $p > \frac{1}{2}$ holds for every direction. In particular for the $(-1, 1)$ direction. About the $\widehat{\text{proj}}_{(-1,1)}$ projection we further know from [19] that there exists a $p' > \frac{3}{8}$, such that for $\frac{3}{8} < p < p'$, $\dim_H(\mathcal{S}_p) > 1$ almost surely conditioned on non-extinction but the $\widehat{\text{proj}}_{(-1,1)}$ projection does not contain any intervals almost surely. Using Theorems 1.9-1.12 we can extend these results. The upper subfigure of Figure 7 summarizes the results derived from [16], [19] and the lower subfigure of the same figure shows our contribution.

- (1) there exists a $\frac{3}{8} < p''$ such that for $\frac{3}{8} < p < p''$, $\dim_H(\mathcal{S}_p) > 1$ almost surely conditioned on non-extinction but the projection $\widehat{\text{proj}}_{(-1,1)}(\mathcal{S}_p)$ has upper box-dimension less than 1.
- (2) For $p > 0.38 \dots = \frac{1}{18^{\frac{1}{3}}}$ (note that this is the same value that occurred in the other projection $\widehat{\text{proj}}_{(1,0)}$ in Figure 6) the projection $\widehat{\text{proj}}_{(-1,1)}(\mathcal{S}_p)$ has positive Lebesgue measure almost surely conditioned on non extinction.
- (3) For $p < \frac{1}{2}$ the projection $\widehat{\text{proj}}_{(-1,1)}(\mathcal{S}_p)$ does not contain an interval almost surely.

The Menger sponge is the 3-dimensional analogue of the Sierpiński carpet and the projections we studied in this chapter are analogous to those three dimensional ones which we denoted by $\text{proj}_{(1,0,0)}$ and $\text{proj}_{(1,1,1)}$.

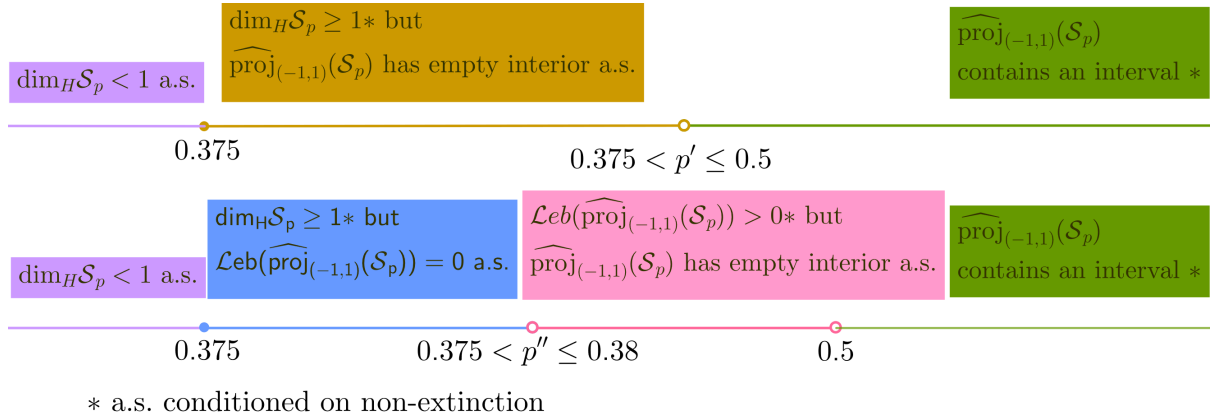


FIGURE 7. Illustration of the extension of our knowledge about the $\widehat{\text{proj}}_{(-1,1)}$ projection of the random Sierpiński carpet.

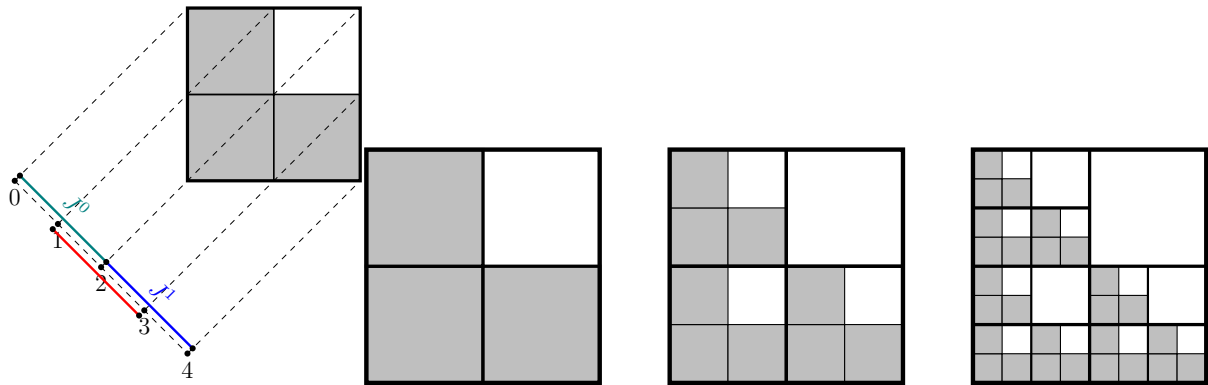


FIGURE 8. The $\widehat{\text{proj}}_{(-1,1)}$ projection of the right-angled Sierpiński gasket.

2.3. Random right-angled Sierpiński gasket. The right angled Sierpiński gasket is the attractor (see Figure 8) of the following self-similar IFS in \mathbb{R}^2 :

$$\mathcal{F} := \left\{ f_i(\mathbf{x}) = \frac{1}{2}(\mathbf{x}) + t_i \right\}_{i=0}^3,$$

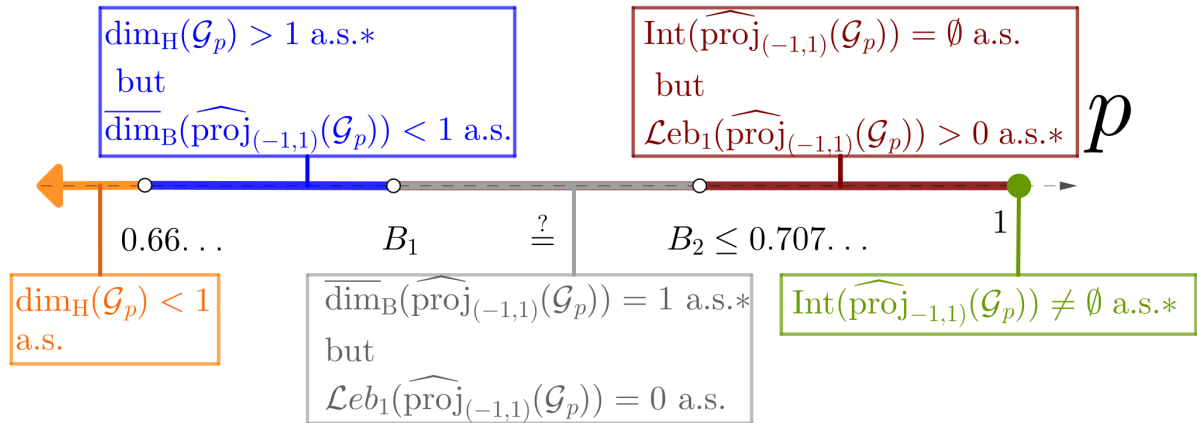
where $\{t_i\}_{i=0}^3$ is an enumeration of the set

$$\left\{ 0, \frac{1}{2} \right\}^2 \setminus \left\{ \left(\frac{1}{2}, \frac{1}{2} \right) \right\}.$$

We obtain the random right-angled Sierpiński gasket by applying the random construction introduced in Definition 1.1 for the deterministic IFS above. We denote the random right-angled Sierpiński gasket with parameter p by \mathcal{G}_p . It is clear that in this case $J^0 = [0, 2]$ and $J^2 = [2, 4]$. The matrices are the following:

$$D_0 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The spectral radii of $p \cdot D_i$ ($i \in \{0, 1\}$) are p . Since the conditions of Theorem 1.9 are satisfied this means that $\text{proj}_{(-1,1)}(\mathcal{G}_p)$ contains an interval if and only if $p = 1$. Its Lebesgue measure is positive when $2p^2 > 1$ i.e. $p > \frac{1}{\sqrt{2}}$, and since the conditions of Theorem 1.12 holds we can find a $p_0 > \frac{2}{3}$, such that if $p \in (\frac{2}{3}, p_0)$, then $\dim_H(\mathcal{G}_p) > 1$



*=conditioned on non-extinction

FIGURE 9. Parameter intervals for the $\widehat{\text{proj}}_{(-1,1)}$ -projection of the random right-angled Sierpiński gasket.

almost surely conditioned on non-extinction even though $\overline{\dim}_B(\widehat{\text{proj}}_{(-1,1)}(\mathcal{G}_p)) < 1$ almost surely.

3. COIN-TOSSING INTEGER SELF-SIMILAR SETS ON THE LINE

We consider the IFS $\mathcal{F} = \{f_i\}_{i=0}^{M-1}$ defined in Example 1.4. We would like to invoke the notation introduced in Definition 1.7. To do so without loss of generality we may assume that the distinct elements of \mathcal{F} are the first m elements of \mathcal{F} . Using them we form a new IFS

$$\mathcal{S} := \left\{ S_i(x) := \frac{1}{L}x + t_i \right\}_{i=0}^{m-1}.$$

Moreover, we may assume, that

$$(3.1) \quad L \geq 2, L \in \mathbb{N}, \text{ and } t_0, \dots, t_m \in \mathbb{Z}, 0 = t_0 < t_1 < \dots < t_{m-1}, L - 1 | t_{m-1}.$$

Recall from (1.5) in Definition 1.7 that $n_j = \#\{f_i \in \mathcal{F} : f_i = S_j\}$. From now on we denote the natural number (see (3.1)) $\tilde{n} := \frac{t_{m-1}}{L-1}$. In this case the interval

$$(3.2) \quad I := [0, \tilde{n} \cdot L]$$

satisfies (1.2).

Lemma 3.1 ([17]).

- (1) For all $\ell \in [M], k \in [N]$ there exist $i \in [N]$ and $b \in [L]$ such that $S_\ell(J^k) = J_b^i$.
- (2) $S_\ell(J^k) = J_b^i$ if and only if $S_\ell(J_{\mathbf{a}}^k) = J_{b\mathbf{a}}^i$ for all $k \geq 0, \mathbf{a} \in [L]^k$.
- (3) If $S_\ell^{-1}(J_{b\mathbf{a}}^i) \neq J_{\mathbf{a}}^k$ holds for all $k \in [N]$ then $\nu(S_\ell^{-1}(J_{b\mathbf{a}}^i)) = 0$.

Corollary 3.2. For all $\ell \in [m]^n$ and $k \in [N]$ there exist an $i \in [N]$ and $\mathbf{a} \in [L]^n$ such that

$$S_\ell(J^k) = J_{\mathbf{a}}^i.$$

Proof. This follows from the first and second part of the previous Lemma 3.1 and induction. \square

Recall that the matrices $\{A_{\mathbf{a}}\}_{\mathbf{a} \in [L]^n}$ were defined in (1.10). It is easy to see that these matrices are well-defined. Namely, for a given ℓ and k , for distinct i_1 and i_2 satisfying

$S_{i_1}(J^k) = J_a^\ell = S_{i_2}(J^k)$, we have $t_{i_1} = t_{i_2}$, which is a contradiction since by definition the translations $\{t_i\}_{i=0,\dots,m-1}$ differ.

Let

$$A := \sum_{a \in [L]} A_a.$$

Lemma 3.3. *For any $n \in \mathbb{N}$, $n \geq 1$, $\mathbf{a} \in [L]^n$ and $\ell, k \in [N]$*

$$A_{\mathbf{a}}(\ell, k) = \# \{(i_1, \dots, i_n) \in [M]^n : f_{i_1, \dots, i_n}(J^k) = J_{\mathbf{a}}^\ell\}.$$

Proof. The assertion follows easily from mathematical induction on n . □

In what follows for an $N \times N$ matrix A ,

$$(3.3) \quad \|A\| := \mathbf{e}^T \cdot A \cdot \mathbf{e}, \quad \mathbf{e}^T = (1, \dots, 1) \in \mathbb{R}^N.$$

Note that we have non-negative matrices, hence the norm defined above in our case coincides with $\|A\|_{1,1}$.

3.1. The ambient probability space. In this section we define coin tossing self-similar sets precisely. Throughout this section we follow the method of Falconer and Jin [10], only we simplify it a little, since the construction we use is much simpler. First let $\widehat{\Omega} := \{0, 1\}^{[M]}$ denote the family of subsets of $[M]$, and let $\widehat{\mathcal{A}}$ be the discrete σ -algebra on it. For $\omega = (\omega_1, \dots, \omega_M) \in \widehat{\Omega}$ and $k = \#\{\ell : \omega_\ell = 1\}$:

$$\widehat{\mathbb{P}}(\{(\omega_1, \dots, \omega_M)\}) := p^k(1-p)^{M-k}.$$

It is easy to see that $\widehat{\mathbb{P}}$ is a probability measure on $(\widehat{\Omega}, \widehat{\mathcal{A}})$. On this space we define the random variable

$$X(\omega) := (X_1(\omega), \dots, X_M(\omega)), \text{ where } X_k(\omega) = \omega_k.$$

For the M -ary tree \mathcal{T} , we define

$$(\Omega, \mathcal{A}, \mathbb{P}) := \bigotimes_{\mathbf{i} \in \mathcal{T}} (\Omega_{\mathbf{i}}, \mathcal{F}_{\mathbf{i}}, \mathcal{P}_{\mathbf{i}}), \text{ where } (\Omega_{\mathbf{i}}, \mathcal{A}_{\mathbf{i}}, \mathcal{P}_{\mathbf{i}}) = (\widehat{\Omega}, \widehat{\mathcal{A}}, \widehat{\mathbb{P}}).$$

For each $\mathbf{i} \in \mathcal{T}$, we define the projection

$$\pi_{\mathbf{i}} : \Omega \rightarrow \Omega_{\mathbf{i}} \text{ by } \pi_{\mathbf{i}}(\boldsymbol{\omega}) := \boldsymbol{\omega}^{\mathbf{i}}.$$

We also define

$$X^{[\mathbf{i}]} := X \circ \pi_{\mathbf{i}}, \text{ i.e. } X_j^{[\mathbf{i}]} = \begin{cases} 1, & \text{if } \omega_j^{\mathbf{i}} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $X^{[\mathbf{i}]}$ are i.i.d random variables with the same distribution as of X . For an $\mathbf{i} = (i_1, \dots, i_n) \in \mathcal{T}$, we define

$$X^{\mathbf{i}}(\boldsymbol{\omega}) := X^{[i_1]}(\boldsymbol{\omega}) \dots X^{[i_n]}(\boldsymbol{\omega}),$$

also the n -th level n -th and eventual survival set:

$$\mathcal{E}_n(\boldsymbol{\omega}) := \{\mathbf{i} \in [M]^n : X^{\mathbf{i}}(\boldsymbol{\omega}) = 1\}, \quad \mathcal{E}_\infty(\boldsymbol{\omega}) := \{\mathbf{i} \in \Sigma^{(M)} : \mathbf{i}|_n \in \mathcal{E}_n(\boldsymbol{\omega}) \forall n \in \mathbb{N}\}.$$

We are given the deterministic self-similar IFS \mathcal{F} and \mathcal{S} as in Definition 1.7 on the line. Put

$$E_n(\boldsymbol{\omega}) := \bigcup_{\mathbf{i} \in \mathcal{E}_n(\boldsymbol{\omega})} I_{\mathbf{i}},$$

where $I_{\mathbf{i}} := f_{\mathbf{i}}(I)$ (recall that I was defined in (3.2)). The coin tossing integer self-similar set on the line corresponding to probability p and the IFS \mathcal{F} is defined as

$$\Lambda_{\mathcal{F}}(p) = \Lambda_{\mathcal{F}}(p, \boldsymbol{\omega}) := \bigcap_{n=1}^{\infty} E_n(\boldsymbol{\omega}).$$

3.2. Proof of Theorem 1.9. The proof uses the method introduced by Dekking and the second author [6]. In that paper the authors consider the sum of two independent one-dimensional Mandelbrot percolations using the two dimensional object which is the product of those Mandelbrot percolations, whereas here we consider a one-dimensional object, which is a generalization of the projection of an any dimensional inhomogeneous Mandelbrot percolation fractal. The product of two Mandelbrot percolation fractals is a random Cantor set which is not a Mandelbrot percolation fractal. Namely in the product case we lose independence, hence statistical self-similarity.

For a $V \in [N]$, we say that the interval J ,

- is of type V if $J = f_{\mathbf{i}}(J^V)$ for some $\mathbf{i} \in \mathcal{E}_n$ for some $n \in \mathbb{N}$;
- is of type V with multiplicity $s \geq 1$ if $\#\{\mathbf{i} \in \mathcal{E}_n : J = f_{\mathbf{i}}(J^V)\} = s$.

For $n \geq 0$; $U, V \in [N]$ and $\mathbf{a} \in [L]^n$, let

$$S^{U,V}(\mathbf{a}) := \{\mathbf{i} \in [M]^n : \mathbf{i} \in \mathcal{E}_n \text{ and } f_{\mathbf{i}}(J^V) = J_{\mathbf{a}}^U\}.$$

Thus, $J_{\mathbf{a}}^U$ is of type V with multiplicity $\#S^{U,V}(\mathbf{a})$. For a $V \in [N]$ let $\{S_U^{U,V}(\mathbf{a})\}_{U \in [N]}$ be independent random variables such that the distribution of $S_U^{U,V}(\mathbf{a})$ is equal to the distribution of $S^{U,V}(\mathbf{a})$, and let

$$S^V(\mathbf{a}) := \bigcup_{U \in [N]} S_U^{U,V}(\mathbf{a}).$$

The following Lemma illuminates the meaning of the previously introduced random variables.

Lemma 3.4. *For any $U, V \in [N]$ and $\mathbf{a} \in [L]^n$:*

$$\mathbb{E}(\#S^{U,V}(\mathbf{a})) = p^n A_{\mathbf{a}}(U, V) \quad \text{and} \quad \mathbb{E}(\#S^V(\mathbf{a})) = p^n C S_{\mathbf{a},V}.$$

Proof. In the deterministic ($p = 1$) case the first part follows from Lemma 3.3. For every level n cylinder the probability of retention is p^n , hence the assertion follows for $p \neq 1$ as well. The second part follows from the following sequence of equalities.

$$\mathbb{E}(\#S^V(\mathbf{a})) = \sum_{U \in [N]} \mathbb{E}(\#S^{U,V}(\mathbf{a})) = p^n \sum_{U \in [N]} A_{\mathbf{a}}(U, V).$$

□

Lemma 3.5. *There exists a $U \in [N]$ and $\mathbf{a} \in [L]^n$ such that*

$$\mathbb{P}(\{\forall V \in [N] : \#S^{U,V}(\mathbf{a}) > 0\}) > 0.$$

Proof. The lemma follows from the second condition of the theorem, namely: under that condition there exist a $U \in [N]$ and $\mathbf{a} \in [L]^n$ for some n such that $A_{\mathbf{a}}(U, V) > 0$ for all $V \in [N]$. The events $\{\#S^{U,V}(\mathbf{a}) > 0\}_{V=0}^{N-1}$ are not exclusive and each has positive probability, hence the probability that all of them happens simultaneously is also positive. □

Fix \tilde{U} and $\tilde{\mathbf{b}}_k \in [L]^k$ in a way that Lemma 3.5 holds for $U = \tilde{U}$ and $\mathbf{a} = \tilde{\mathbf{b}}_k$. Let

$$(3.4) \quad \gamma_V := p \cdot \min_{\mathbf{a} \in [L]} C S_{\mathbf{a},V} \quad \text{and} \quad \gamma := \min_{V \in [N]} \gamma_V.$$

Note that $\gamma > 1$ is equivalent to the first condition of Theorem 1.9. That is,

$$(3.5) \quad \gamma > 1 \iff p \cdot CS_{a,j} > 1 \text{ for all } a \in [L] \text{ and } j \in [N].$$

Let H^0 be the event in Lemma 3.5, namely that the interval $J_{\mathbf{b}_k}^{\tilde{U}}$ is of every type, i.e.

$$H^0 := \left\{ \forall V \in [N] : S^{\tilde{U},V}(\tilde{\mathbf{b}}_k) > 0 \right\}.$$

Then by Lemma 3.5,

$$p_0 := \mathbb{P}(H^0) > 0.$$

This means that with positive probability we can find

$$(3.6) \quad \mathbf{i}^{0,0}, \dots, \mathbf{i}^{0,N-1} \in [M]^k \text{ such that } f_{\mathbf{i}^{0,V}}(J^V) = J_{\tilde{\mathbf{b}}_k}^{\tilde{U}}.$$

We say that $\mathbf{i} \in [M]^*$ makes $J_{\mathbf{a}}^U$ ($U \in [N]$, $\mathbf{a} \in [L]^{|i|}$) a type V ($V \in [N]$) interval if the following two hold:

- $\mathbf{i} \in \mathcal{E}_{|i|}$ and
- $f_{\mathbf{i}}(J^V) = J_{\mathbf{a}}^U$.

Hence (3.6) can be phrased as we can find indices that make $J_{\tilde{\mathbf{b}}_k}^{\tilde{U}}$ of every type.

In what follows we define several stochastic processes counting the multiplicity of different types of the intervals of the form $J_{\mathbf{a}_n}^V$ for $\mathbf{a}_n = a_1, \dots, a_n \in [L]^n$ and $V \in [N]$ for the different values of $n \in \mathbb{N}$. We apply large deviation theory to prove that these processes simultaneously do not die out with positive probability, implying that the random attractor contains an interval with positive probability. Lastly a standard argument reveals (see Lemma 3.9) that this is a 0 – 1 event conditioned on non-extinction.

First we define a process that collects the different sets of retained indices $\{\mathbf{i}_V \in [M]^*\}_{V \in [N]}$, such that the elements of the sets make an interval of all types. Start the process with the 0-th level N -tuple (see (3.6)),

$$\mathfrak{T}^0(\emptyset) := \left\{ (\mathbf{i}^{0,0}, \dots, \mathbf{i}^{0,N-1}) \right\}.$$

For $\mathbf{c}_n \in [L]^n$ the level- n collection of N -tuples of \mathbf{c}_n is,

$$\mathfrak{T}^n(\mathbf{c}_n) = \left\{ \left(\mathbf{i}_0^{n,0}, \dots, \mathbf{i}_0^{n,N-1} \right), \dots, \left(\mathbf{i}_{j-1}^{n,0}, \dots, \mathbf{i}_{j-1}^{n,N-1} \right) \right\},$$

in a way that the following three conditions hold:

- $\mathbf{i}_\ell^{n,V} \in \mathcal{E}_{n+k}$ for all $V \in [N]$ and $\ell \in [j]$;
- $f_{\mathbf{i}_k^{n,V}}(J^V) = J_{\tilde{\mathbf{b}}_k \mathbf{c}_n}^{\tilde{U}}$, meaning that $\mathbf{i}_\ell^{n,V}$ makes $J_{\tilde{\mathbf{b}}_k \mathbf{c}_n}^{\tilde{U}}$ be of type V ;
- for all $\ell_1, \ell_2 \in [j]$ and for all $V \in [N]$: $\mathbf{i}_{\ell_1}^{n,V} \neq \mathbf{i}_{\ell_2}^{n,V}$, meaning that all elements appear only once.

Observe that the distribution of $\#\mathfrak{T}^n(\mathbf{a}_n) | H^0$ is the same as the distribution of $\min_{U \in [N]} \#S^U(\mathbf{a}_n)$.

Now, given H^0 consider $\#\mathfrak{T}^n(\mathbf{a}_n)$. Recall that $\gamma = \min_{V \in [N]} \min_{a \in [L]} \mathbb{E}(\#S^V(a))$ which by the first assumption of the Theorem is greater than 1 (see (3.5)). Choose ρ such that $1 < \rho < \gamma$, and define the events

$$H_n := \left\{ \forall \mathbf{a}_n \in [L]^n : \#\mathfrak{T}^n(\mathbf{a}_n) > \rho^n \right\}.$$

As usual for an event E let \bar{E} denote its complement.

Lemma 3.6. *There exists a $0 < \delta < 1$ such that for all $n \in \mathbb{N}$*

$$\mathbb{P}(\bar{H}_{n+1} | H_n, H_0) \leq L^{n+1} \cdot N \cdot \delta^{\rho^n}.$$

Fact 3.7 (Azuma-Hoeffding inequality). *Let $Z_0^U(k), \dots, Z_\ell^U(k)$ be i.i.d random variables distributed according to $\#S^U(k)$. Then*

$$\mathbb{P} \left(Z_0^U(k) + \dots + Z_\ell^U(k) \leq \ell \cdot \rho \right) \leq \delta^\ell, \text{ for some } 0 < \delta < 1.$$

Proof of Lemma 3.6.

$$\mathbb{P} \left(\overline{H_{n+1}} | H_n, H_0 \right) \leq \sum_{\mathbf{a}_n \in [L]^n} \sum_{a \in [L]} \mathbb{P} \left(\#\mathfrak{T}^n(\mathbf{a}_n a) < \rho^{n+1} \mid \forall \mathbf{c}_n \in [L]^n : \#\mathfrak{T}^n(\mathbf{c}_n) \geq \rho^n \right).$$

For a fixed $\mathbf{a}_n \in [L]^n$ and $a \in [L]$

$$\begin{aligned} \mathbb{P} \left(\#\mathfrak{T}^n(\mathbf{a}_n a) < \rho^{n+1} \mid \forall \mathbf{c}_n \in [L]^n : \#\mathfrak{T}^n(\mathbf{c}_n) \geq \rho^n \right) \\ \leq \sum_{U \in [N]} \mathbb{P} \left(\#S^U(\mathbf{a}_n a) < \rho^{n+1} \mid \min_{V \in [N]} \#S^V(\mathbf{a}_n) \geq \rho^n \right). \end{aligned}$$

For any fixed U we can use Fact 3.7 to upper-bound the last probability. This is because conditioned on the event $\left\{ \min_{V \in [N]} \#S^V(\mathbf{a}_n) \geq \rho^n \right\}$ we have at least ρ^n level- n N -tuples in the L -adic interval \mathbf{a}_n , hence $\#S^U(\mathbf{a}_n a)$ is bounded below by the sum of at least ρ^n independent random variables, distributed according to $\#S^U(a)$. The random variables in this sum are independent because by the construction, given \mathcal{E}_n the events that we retain or discard different cylinders on level $n+1$ are independent, hence the random variables in the sum are also independent. Thus

$$\mathbb{P} \left(\#S^U(\mathbf{a}_n a) < \rho^{n+1} \mid \min_{V \in [N]} \#S^V(\mathbf{a}_n) \geq \rho^n \right) \leq \delta(U, a)^{\rho^n}$$

for some $0 < \delta(U, a) < 1$ and for any particular $U \in [N]$, $\mathbf{a}_n \in [L]^n$ and $a \in [L]$.

Choose $\delta := \max_{U \in [N]} \max_{a \in [L]} \delta(U, a) < 1$, in this way we get that

$$(3.7) \quad \mathbb{P} \left(\overline{H_{n+1}} \mid H_n \right) \leq \sum_{\mathbf{a}_n \in [L]^n} \sum_{a \in [L]} \sum_{U \in [N]} \delta^{\rho^n} = L^{n+1} \cdot N \cdot \delta^{\rho^n}.$$

□

Lemma 3.8. *For every $n \in \mathbb{N}$:*

$$\mathbb{P} \left(\forall V \in [N], \forall \mathbf{a}_n \in [L]^n : \#S^V(\mathbf{a}_n) \geq \rho^n \right) > 0.$$

Proof. This is an adaptation of Dekking and Simon ([6, Lemma 1]). For $\mathbf{a}_n \in [L]^n$:

$$\mathbf{e}^T A_{\mathbf{a}_n} \geq p^n \mathbf{e}^T A_{\mathbf{a}_n} = [CS_{\mathbf{a}_n, 0}, \dots, CS_{\mathbf{a}_n, N-1}] \geq [\rho^n, \dots, \rho^n].$$

This means that in the deterministic setup we have enough indices for the event to happen for any \mathbf{a}_n . Hence it follows that $\#S^V(\mathbf{a}_n)$ happens with positive probability for each \mathbf{a}_n and V . □

In what follows we prove that

$$(3.8) \quad \mathbb{P} \left(\forall n, \forall \mathbf{a}_n : \#\mathfrak{T}^n(\mathbf{a}_n) > 0 \right) > 0.$$

This proves that $\Lambda_{\mathcal{F}}(p)$ contains an interval with positive probability, because (3.8) means that with positive probability for any n we retain something in every sub-interval $J_{\mathbf{b}_k \mathbf{a}_n}^{\tilde{U}}$ (for all $\mathbf{a}_n \in [L]^n$) of $J_{\mathbf{b}_k}^{\tilde{U}}$. The inequality in (3.8) holds since

$$\mathbb{P} \left(H_n, n > r \right) = \mathbb{P} \left(H_0 \right) \cdot \mathbb{P} \left(H_r \right) \prod_{n=r}^{\infty} \mathbb{P} \left(H_{n+1} \mid H_n \right) \geq p_0 \cdot \mathbb{P} \left(H_r \right) \prod_{n=r}^{\infty} \left(1 - L^{n+1} \cdot N \cdot \delta^{\rho^n} \right).$$

By Lemma 3.5 p_0 is positive, $\mathbb{P}(H_r)$ is positive by Lemma 3.8, and we can choose r such that the last expression is positive. Which finishes the proof of

$$(3.9) \quad \mathbb{P}(\Lambda_{\mathcal{F}}(p) \text{ contains an interval}) > 0.$$

Lemma 3.9. *Let θ be a possible property of $\Lambda_{\mathcal{F}}(p)$. Let \mathfrak{A} denote the event that θ holds for $\Lambda_{\mathcal{F}}(p)$. Assume that \mathfrak{A} has the following properties:*

- (1) *happens almost surely if the process dies out,*
- (2) *happens if and only if θ holds for every intersection of the set with level n cylinders, namely for every n and i_1, \dots, i_n : θ holds for $\Lambda_{\mathcal{F}}(p) \cap I_{i_1, \dots, i_n}$ and*
- (3) *is not a sure event.*

Then conditioned on non-extinction \mathfrak{A} almost surely does not happen.

Proof. The proof is based on a standard argument (similar to the one in [9, page 471]) using statistical self-similarity. Let $\mathfrak{C} := \{\#\mathcal{E}_\infty > 0\}$ denote the event that the process does not die out. By the third assumption of the lemma $\mathbb{P}(\mathfrak{A}) = \varepsilon < 1$. From the theory of branching processes we know that conditioned on non-extinction the number of retained level- n cylinders tends to infinity almost surely, i.e. $\mathbb{P}(\#\mathcal{E}_n < \widetilde{M}^n \mid \mathfrak{C}) \rightarrow 0$ as $n \rightarrow \infty$ for all \widetilde{M} . Now, by the assumption of the lemma:

$$\begin{aligned} \mathbb{P}(\mathfrak{A} \mid \mathfrak{C}) &= \mathbb{P}(\mathfrak{A} \mid \#\mathcal{E}_n \geq \widetilde{M}) \cdot \mathbb{P}(\#\mathcal{E}_n \geq \widetilde{M} \mid \mathfrak{C}) + \mathbb{P}(\mathfrak{A} \mid \#\mathcal{E}_n < \widetilde{M}) \cdot \mathbb{P}(\#\mathcal{E}_n < \widetilde{M} \mid \mathfrak{C}) \\ &\leq \varepsilon^{\widetilde{M}} + \mathbb{P}(\#\mathcal{E}_n < \widetilde{M} \mid \mathfrak{C}). \end{aligned}$$

The inequality follows from statistical self-similarity of $\Lambda_{\mathcal{F}}(p)$ and the second condition of the lemma.

First letting $n \rightarrow \infty$, then $\widetilde{M} \rightarrow \infty$ gives that

$$\mathbb{P}(\mathfrak{A} \mid \mathfrak{C}) = 0.$$

□

Proof of Theorem 1.9. Follows from (3.9) and Lemma 3.9, using that the property of the random attractor that *its projection does not contains an interval* satisfies the three assumptions of Lemma 3.9.

□

3.3. Proof of Theorem 1.10.

Proof of Theorem 1.10. Parts of the following proof resemble [6, Proof of Theorem 1 (b)]. Let $a \in [L]$ such that the spectral radius of $p \cdot A_a$ is smaller than 1. Let \mathbf{a}^n denote the n -vector consisting only of a , i.e.

$$\bar{\mathbf{a}}^n := (a, \dots, a).$$

It is a well-known fact that if the spectral radius of an $N \times N$ matrix B is smaller than 1, then

$$\lim_{n \rightarrow \infty} \|B^n\| = 0.$$

This implies by the assumptions of the theorem that $\lim_{n \rightarrow \infty} \|(p \cdot A_a)^n\| = \lim_{n \rightarrow \infty} \|p^n \cdot A_{\bar{\mathbf{a}}^n}\| = 0$. By the sub-multiplicativity of the matrix norm defined in (3.3), it follows that for any $\mathbf{c}_k = (c_1, \dots, c_k) \in [L]^j$:

$$(3.10) \quad \lim_{n \rightarrow \infty} \|p^{n+k} A_{\mathbf{c}_k \bar{\mathbf{a}}^n}\| \leq \lim_{n \rightarrow \infty} \|p^k A_{\mathbf{c}_k}\| \cdot \|p^n A_{\bar{\mathbf{a}}^n}\| = \|p^k A_{\mathbf{c}_k}\| \lim_{n \rightarrow \infty} \|p^n A_{\bar{\mathbf{a}}^n}\| = 0.$$

Let $\mathbf{c}_k \in [L]^k$ be given and Z_n denote the number of level $k+n$ cylinders intersecting $\cup_{j \in [N]} J_{\mathbf{c}_k \bar{\mathbf{a}}^n}^j$. From Lemma 3.4 and (3.10) we know that

$$\mathbb{E}(Z_n) = \|p^{k+n} A_{\mathbf{c}_k \bar{\mathbf{a}}^n}\| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

thus by Markov's inequality:

$$\mathbb{P}(Z_n \geq 1) \leq \mathbb{E}(Z_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In this way the points

$$\bigcup_{j \in [N]} \bigcap_{n \rightarrow \infty} J_{\mathbf{c}_k \bar{\mathbf{a}}^n}^j$$

are not contained in $\Lambda_{\mathcal{F}}(p)$ with probability one. By varying \mathbf{c}_k we get a countable dense set which is not contained in $\Lambda_{\mathcal{F}}(p)$ with probability one, hence it can not contain an interval. \square

3.4. Proof of Theorem 1.11.

Lemma 3.10. *If there exist a positive measure set K such that for $x \in K$: $\mathbb{P}(x \in \Lambda_{\mathcal{F}}(p)) > 0$, then*

$$(3.11) \quad \mathbb{P}(\mathcal{L}eb_1(\Lambda_{\mathcal{F}}(p)) > 0) > 0.$$

Proof. 3.11 holds if and only if $\mathbb{E}(\mathcal{L}eb_1(\Lambda_{\mathcal{F}}(p))) > 0$. Observe that

$$\begin{aligned} \mathbb{E}(\mathcal{L}eb_1(\Lambda_{\mathcal{F}}(p))) &= \int_{\Omega} \mathcal{L}eb_1(\Lambda_{\mathcal{F}}(p, \boldsymbol{\omega})) d\mathbb{P}(\boldsymbol{\omega}) = \int_{\Omega} \int_I \mathbb{1}\{x \in \Lambda_{\mathcal{F}}(p, \boldsymbol{\omega})\} d\mathcal{L}eb_1(x) d\mathbb{P}(\boldsymbol{\omega}) \\ &= \int_I \int_{\Omega} \mathbb{1}\{x \in \Lambda_{\mathcal{F}}(p, \boldsymbol{\omega})\} d\mathbb{P}(\boldsymbol{\omega}) d\mathcal{L}eb_1(x) = \int_I \mathbb{P}(x \in \Lambda_{\mathcal{F}}(p)) d\mathcal{L}eb_1(x). \end{aligned}$$

Since $\mathbb{P}(x \in \Lambda_{\mathcal{F}}(p)) > 0$ on a positive measure set, this is positive. \square

It follows from the Lemma above combined with Lemma 3.9 that the statement of Theorem 1.11 follows if we prove the following Proposition.

Proposition 3.11. *Under the conditions of Theorem 1.11 there exists a set K of positive $\mathcal{L}eb_1$ measure such that*

$$\mathbb{P}(x \in \Lambda_{\mathcal{F}}(p)) > 0 \quad \text{for } \mathcal{L}eb_1\text{-a.e. } x \in K.$$

Proof of Proposition 3.11. We will use the first part of the proof of Theorem 1.9, namely the part until (3.6). We choose \tilde{U} and $\tilde{\mathbf{b}}_k \in [L]^k$ in a way that Lemma 3.5 holds for $U = \tilde{U}$ and $\mathbf{a} = \tilde{\mathbf{b}}_k$. Let

$$K := J_{\tilde{\mathbf{b}}_k}^{\tilde{U}}.$$

$U \sim \text{Uniform}(K)$ and \mathfrak{P} and \mathfrak{E} denote the corresponding distribution and expectation respectively. In what follows we will prove that

$$\mathfrak{P}(\mathbb{P}(U \in K) > 0) = 1,$$

because this is equivalent to the statement of Proposition 3.11 which we want to prove. For the proof of 3.4 we will use the theory of branching processes in random environment.

We begin by defining a process, it follows from some of the results of [1] on branching processes in random environments that under some conditions 3.4 holds. We finish the section by proving that the process we defined satisfies the conditions.

Namely, for a given $\mathbf{a} = (a_1, \dots, a_n, \dots) \in \Sigma^{(L)}$ let

$$\tilde{\mathfrak{I}}^0(\mathbf{a}) := (\mathbf{i}^{0,0}, \dots, \mathbf{i}^{0,N-1}),$$

where $\mathbf{i}^{0,0}, \dots, \mathbf{i}^{0,N-1}$ was defined in (3.6). Lemma 3.5 guarantees that such $\mathbf{i}^{0,0}, \dots, \mathbf{i}^{0,N-1}$ exists, since the second condition of Theorem 1.11 is stronger than the second assumption of Theorem 1.9. Recursively, if we have

$$\tilde{\mathfrak{I}}^{t-1}(\mathbf{a}) = \left\{ \left(\mathbf{i}_1^{t-1,0}, \dots, \mathbf{i}_1^{t-1,N-1} \right), \dots, \left(\mathbf{i}_s^{t-1,0}, \dots, \mathbf{i}_s^{t-1,N-1} \right) \right\},$$

then $(\mathbf{i}^{t,0}, \dots, \mathbf{i}^{t,N-1})$ with elements $\mathbf{i}^{t,V}$ is in $\tilde{\mathfrak{I}}^t(\mathbf{a})$ if and only if, for a Δ natural number to be defined later the following hold:

- $\mathbf{i}^{t,V} \in \mathcal{E}_{\Delta \cdot t + k}$
- $\mathbf{i}^{t,V}|_{\Delta \cdot (t-1) + k} \in \tilde{\mathfrak{I}}^{t-1}(\mathbf{a})$
- $f_{\mathbf{i}^{t,V}}(J^V) = J_{\tilde{b}_k \mathbf{a}|_{t \cdot \Delta}}^{\tilde{U}}$
- If $\mathbf{i}^{t,V}$ is in an element of $\tilde{\mathfrak{I}}^t$, then it is not contained in another element of $\tilde{\mathfrak{I}}^t$.

The elements of $\tilde{\mathfrak{I}}^n(\bar{\theta})$ of the form $(\mathbf{i}^{n,0}, \dots, \mathbf{i}^{n,N-1})$ are called level- n N -tuples.

In what follows we define the final process denoted by $\mathcal{Z} = \{\mathcal{Z}_n\}_{n \in \mathbb{N}}$ that we use for the proof. This process is a branching process in a random (i.i.d.) environment. This can be thought of as process made up of two steps. The first step is to choose the environment $\bar{\theta}$ and then consider the branching process in the varying environment $\bar{\theta}$. Usually (see for example [13], [1]) the environment is defined in terms of probability generating functions determining the distribution in the given environment, although here, to simplify the notations, we consider the indices of the probability generating functions instead, which has a one-to-one correspondence with the probability generating functions.

The first step is hence to choose the environment. First for each $i \in \mathbb{N}$ we choose a_i uniformly from $[L]$ independently of each other. But instead of considering $\mathbf{a} = (a_1, a_2, \dots)$ we consider a "multiple step environment",

$$(3.12) \quad \bar{\theta} = (\theta_0, \dots, \theta_n, \dots), \text{ where } \theta_k = (a_{k\Delta+1}, \dots, a_{(k+1)\Delta}).$$

It is clear that this is equivalent of choosing θ_k uniformly and independently for each k from the set $[L]^\Delta$. This way, the branching process in the already chosen varying environment $\bar{\theta}$ is,

$$\begin{aligned} \mathcal{Z}_0(\bar{\theta}) &:= \#\tilde{\mathfrak{I}}^0(\mathbf{a}) = 1, \\ \mathcal{Z}_n(\bar{\theta}) &:= \#\tilde{\mathfrak{I}}^n(\mathbf{a}). \end{aligned}$$

It is clear that the above defined $\mathcal{Z}_n(\bar{\theta})$ process satisfies the conditions of a branching process in a varying environment. This is because

$$\mathcal{Z}_{n+1}(\bar{\theta}) = \sum_{i=1}^{\mathcal{Z}_n(\bar{\theta})} X_{n,i}(\bar{\theta}),$$

where $X_{n,i}(\bar{\theta})$ is the number of level- $(n+1)$ N -tuples coming from the i -th level- n N -tuple in $\tilde{\mathfrak{I}}^n(\mathbf{a})$. The random variables $\{X_{n,i}(\bar{\theta})\}_{i=1}^{\mathcal{Z}_n(\bar{\theta})}$ are independent because what happens in different retained cylinders are independent of each other.

If $\mathcal{Z}_n(\bar{\theta})$ does not die out for a given $\bar{\theta} = (a_1, \dots, a_{k\Delta+1}, \dots)$, then conditioned on H^0 the point x which has L -adic expansion $\tilde{\mathbf{b}}_k, a_1, \dots, a_{k\Delta+1}, \dots$ shifted with the left endpoint of the interval $J^{\tilde{U}} = [\hat{u}_1, \hat{u}_2]$ i.e. $x = \hat{u}_1 + \sum_{j=1}^k \hat{b}_{k_j} L^{j-1} + \sum_{j=1}^{\infty} a_j L^{j-1+k}$ is contained in $\Lambda_{\mathcal{F}}(p)$. This is because if the process does not die out then on every level the cylinder containing x is retained, since it is of a retained type.

Before we could consider the conditions under which the process we defined does not die out, we prove a useful fact that we will use to prove that our \mathcal{Z} process satisfies the conditions that we will define soon.

Denote the probability that the interval J_a^V is of every type by $q(a, V)$, namely

$$q(a, V) := \mathbb{P}(\forall U \in [N] \exists i \in \mathcal{E}_1 : f_i(J^U) = J_a^V) \text{ and } q := \min_{a \in [L]} \max_{V \in [N]} q(a, V).$$

Fact 3.12. $q > 0$.

Proof of the Fact. Every matrix A_a has a positive row $A_{a,V}$ by the second condition of Theorem 1.11. Thus,

$$\mathbb{P}(\{\forall U : \#S^{V,U}(a) > 0\}) = \mathbb{P}\left(\prod_{U \in [N]} \#S^{V,U}(a) > 0\right) > 0 \text{ iff } \mathbb{E}\left(\prod_{U \in [N]} \#S^{V,U}(a)\right) > 0,$$

which is by independence and Lemma 3.4 equivalent to

$$\prod_{U \in [N]} \mathbb{E}(\#S^{V,U}(a)) = \prod_{U \in [N]} p \cdot A_a(V, U) > 0.$$

The last inequality holds by the second condition of Theorem 1.11. \square

Let $V \in [N]$ be arbitrary such that $V = \max_{W \in [N]} q(a, W)$. Then we define

$$U(a) := V.$$

Using [1, Theorem 3], under conditions **C1** and **C2** below, \mathfrak{P} -almost every $\bar{\theta}$ the process $\mathcal{Z}_n(\bar{\theta})_{n \in \mathbb{N}}$ does not die out with positive probability. Observe that this is equivalent to (3.4), which we wanted to prove in order to show that Proposition 3.11 holds. In what follows we define the conditions and show that the process we defined satisfies these conditions.

C1: There exists a $c > 0$ such that for all $\bar{\theta}$: $\mathbb{P}(\mathcal{Z}_1(\bar{\theta}) > 0) > c$;

C2: $\frac{1}{L^\Delta} \sum_{\substack{(a_1, \dots, a_\Delta) \\ \in [L]^\Delta}} \log[\mathbb{E}(\mathcal{Z}_1(\bar{\theta}) | \theta_0 = (a_0, \dots, a_\Delta))] > 0$,

where Δ is a natural number which is to be defined later (it was introduced as the length of the elements of the environmental random variable, see (3.12)). The following proof is a straightforward modification of [15, Proof of Lemma 1.] and its consequences. To see that Condition **C1** holds, note that by the definition of q , for some $\varepsilon \in (0, 1)$ we have

$$\mathbb{P}(\mathcal{Z}_1(\bar{\theta}) > 0) > p_0 \cdot q^\Delta (1 - \varepsilon).$$

Now we prove that condition **C2** holds as well. For this we first prove a useful Lemma. Recall, that

$$g_j = \left(\prod_{a \in [L]} CS_{a,j} \right)^{\frac{1}{L}}$$

and denote $\hat{\Gamma} = \min_{j \in [N]} g_j$. By the first assumption of Theorem 1.11 $p \cdot \hat{\Gamma} > 1$.

Lemma 3.13. *For any $n \in \mathbb{N}$ and $V \in [N]$:*

$$\mathfrak{E}[\log(\mathbb{E}(\#S^V(a_1, \dots, a_n)))] \geq n \cdot \log[\hat{\Gamma} \cdot p].$$

Proof of Lemma. The proof is by induction, namely, for $n = 1$:

$$\begin{aligned} \mathfrak{E} [\log (\mathbb{E} (\#S^V (a)))] &= \frac{1}{L} \sum_{a \in [L]} \log (\mathbb{E} (\#S^V (a))) = \frac{1}{L} \log \left(\prod_{i \in [L]} p \cdot CS_{i,V} \right) \\ &= \log (p \cdot g_V) \geq \log (p \cdot \widehat{\Gamma}). \end{aligned}$$

Assume that $\mathfrak{E} [\log (\mathbb{E} (\#S^V (a_1, \dots, a_n)))] \geq n \cdot \log [\widehat{\Gamma} \cdot p]$. Let $\mathbf{a} := (a_1, \dots, a_{n+1})$ and $\mathbf{a}|_n := (a_1, \dots, a_n)$. From $\mathbb{E} (\#S^{U,V} (a)) = p \cdot A_a (U, V)$, it follows, that

$$\begin{aligned} \mathbb{E} (\#S^V (\mathbf{a})) &= \sum_{U \in [N]} \mathbb{E} (\#S^{U,V} (a_{n+1})) \cdot \mathbb{E} (\#S^U (\mathbf{a}|_n)) \\ &= \sum_{U \in [N]} \left[\frac{A_{a_{n+1}} (U, V)}{CS_{a_{n+1},V}} \right] \cdot [p \cdot CS_{a_{n+1},V} \mathbb{E} (\#S^U (\mathbf{a}|_n))]. \end{aligned}$$

By the concavity of the log function

$$\mathbb{E} (\#S^V (\mathbf{a})) \geq \log (p \cdot CS_{a_{n+1},V}) + \sum_{U \in [N]} \frac{A_{a_{n+1}} (U, V)}{CS_{a_{n+1},V}} \log [\mathbb{E} (\#S^U (\mathbf{a}|_n))].$$

Hence by independence and the linearity of expectation

$$\begin{aligned} \mathfrak{E} [\log (\mathbb{E} (\#S^V (\mathbf{a})))] &= \mathfrak{E} [\log (p \cdot CS_{a_{n+1},V})] \\ &\quad + \sum_{U \in [N]} \mathfrak{E} \left[\frac{A_{a_{n+1}} (U, V)}{CS_{a_{n+1},V}} \right] \mathfrak{E} [\log (\mathbb{E} (\#S^U (\mathbf{a}|_n)))] . \end{aligned}$$

The first part of the sum is

$$\mathfrak{E} [\log (p \cdot CS_{a_{n+1},V})] = \log (g_V \cdot p) \geq \log (\widehat{\Gamma} \cdot p),$$

and the second, by the induction hypothesis is

$$\mathfrak{E} [\log (\mathbb{E} (\#S^U (\mathbf{a}|_n)))] \geq n \cdot \log (\widehat{\Gamma} \cdot p).$$

Hence

$$\begin{aligned} \sum_{U \in [N]} \mathfrak{E} \left[\frac{A_{a_{n+1}} (U, V)}{CS_{a_{n+1},V}} \right] \mathfrak{E} [\log (\mathbb{E} (\#S^U (\mathbf{a}|_n)))] &= n \cdot \log (\widehat{\Gamma} \cdot p) \sum_{U \in [N]} \mathfrak{E} \left[\frac{A_{a_{n+1}} (U, V)}{CS_{a_{n+1},V}} \right] \\ &= n \cdot \log (\widehat{\Gamma} \cdot p) \end{aligned}$$

combining the two above gives the required inequality. \square

It follows from

$$\begin{aligned} \mathbb{E} (\mathcal{Z}_1 (\bar{\theta}) | \theta_0 = (a_1, \dots, a_\Delta)) &\geq \mathbb{E} (\#S^{U(a_\Delta)} (a_1, \dots, a_\Delta)) q (a_\Delta, U (a_\Delta)) \\ &\geq q \cdot \mathbb{E} (\#S^{U(a_\Delta)} (a_1, \dots, a_\Delta)), \end{aligned}$$

that

$$\begin{aligned} \mathfrak{E} [\log (\mathbb{E} (\mathcal{Z}_1 (\bar{\theta}) | \theta_0 = (a_1, \dots, a_\Delta)))] &\geq \mathfrak{E} [\log (\mathbb{E} (\#S^V (a_1, \dots, a_\Delta)))] \\ &\quad + \log (q) \geq \Delta \cdot \log (\widehat{\Gamma} \cdot p) + \log (q). \end{aligned}$$

Since $\log(\widehat{\Gamma} \cdot p) > 0$ we can choose Δ , such that

$$\Delta \cdot \log(\widehat{\Gamma} \cdot p) + \log(q) > 0.$$

□

This completes the proof of Proposition 3.11.

3.5. Proof of Theorem 1.12. In what follows we introduce the pressure function for a set of matrices, and we state a crucial Lemma (Lemma 3.15) about it. This Lemma is proven in a special case when the matrices are corresponding to sets that are rational projections of two-dimensional carpets in [3]. Our setup (where we instead consider the one-dimensional special case of the systems occurring in [17]) is more general than that, hence we present the proof of the Lemma in the Appendix even though the proofs in [3] in addition to some from [17] can almost entirely be used in this more general setup. Using the pressure function we will be able to upper bound the exponential growth rate of the expected number of cylinders required to cover the approximations of our set.

Recall that we defined the matrices A_0, \dots, A_{L-1} in 1.10.

Definition 3.14 (Pressure function). *For the set of matrices A_0, \dots, A_{L-1} we define the pressure function in the following way:*

$$(3.13) \quad P(t) = \frac{1}{\log L} \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\mathbf{a} \in [L]^n} \|A_{a_1, \dots, a_n}\|^t.$$

The Legendre-transform of the pressure function $P(t)$ is denoted by $P^*(\delta)$, i.e.

$$(3.14) \quad P^*(\delta) = \inf_t \{-\delta t + P(t)\}.$$

For the proof of Theorem 1.12 we will use the following Lemma, the proof of which can be found in the Appendix.

Proposition 3.15. *There exists an $\alpha < \frac{\log M}{\log L} - 1$ such that*

$$P^*(\alpha) < 1.$$

3.5.1. Counting process of the L -adic intervals intersecting the Mandelbrot percolation fractal. Throughout this section we will always condition on the event of non-extinction of the process $\#\mathcal{E}_n$. Let $\mathbf{Y} = (Y_0, Y_1, \dots)$ denote the counting process of the level- n L -adic intervals intersected with the E_n , the n -th level approximation of the coin tossing self-similar set, namely:

$$\begin{aligned} Y_0 &:= 1 \\ Y_n &:= \#\{j\mathbf{a} \in [N] \times [L]^n : J_{\mathbf{a}}^j \cap E_n \neq \emptyset\}. \end{aligned}$$

With this notation

$$(3.15) \quad \overline{\dim}_B(\Lambda_{\mathcal{F}}(p)) \leq \limsup_{n \rightarrow \infty} \frac{\log Y_n}{n \log L}$$

Lemma 3.16 (Fekete's lemma). *If b_n is superadditive sequence i.e. $b_{n+m} \geq b_n + b_m$, then there exists $\lim_{n \rightarrow \infty} \frac{b_n}{n}$ and $\lim_{n \rightarrow \infty} \frac{b_n}{n} = \sup_n \frac{b_n}{n}$.*

Corollary 3.17. *Consider the strictly positive sequence a_n . Assume that there exist a k such that for all n and m , $a_{n+m} \geq \frac{a_n \cdot a_m}{k}$. Then the limit $\lim_{n \rightarrow \infty} \frac{1}{n} \log a_n$ exist and $\lim_{n \rightarrow \infty} \frac{1}{n} \log a_n = \sup_n \frac{1}{n} \log a_n$.*

Proof. The proof is straightforward using Fekete's lemma for the sequence $b_n = \log(a_n) - \log(k)$. \square

Definition 3.18 (Quasi supermultiplicative sequence). *If a_n is a sequence as in Corollary 3.17, then we call a_n quasi supermultiplicative.*

Since we conditioned on non-extinction of $\#\mathcal{E}_n$ it is true that for each n $Y_n > 0$ and $\mathbb{E}(Y_n) > 0$. In what follows we will show that the sequence $\{\mathbb{E}(Y_n)\}_n$ is a quasi supermultiplicative positive sequence, hence by 3.17 there exist a number

Lemma 3.19. *There exists a λ such that*

$$(3.16) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}Y_n =: \lambda,$$

and

$$\lambda = \sup_n \frac{1}{n} \log \mathbb{E}Y_n.$$

Proof. First we will prove that the sequence $\mathcal{E}(Y_n)$ is quasi supermultiplicative, namely that there exists a constant C such that for every $n, m > 0$

$$\mathbb{E}(Y_{n+m}) \geq \frac{\mathbb{E}(Y_n)\mathbb{E}(Y_m)}{C}.$$

Then the assertion follows from Lemma 3.17.

Assume that we are at level n , then the n -th approximation of the coin tossing self-similar set intersects Y_n L -adic intervals. Denote these intervals K_1, \dots, K_{Y_n} . It is clear that $Y_n \neq \#\mathcal{E}_n$. For each of the intervals K_j we can find at least one $\mathbf{i} \in \mathcal{E}_n$ such that $K_j \subset f_{\mathbf{i}}(I)$. If $\mathbf{i} \in \mathcal{E}_n$ then $f_{\mathbf{i}}(I)$ will intersect exactly N level- n L -adic intervals, hence intersecting cylinders can occupy only $N + 2(N - 1) < 3N$ intervals of K_1, \dots, K_{Y_n} which means that we can choose a subset $\tilde{\mathcal{E}}_n \subset \mathcal{E}_n$ with more than $\lfloor \frac{Y_n}{3N} \rfloor$ elements such that if $\mathbf{i}, \mathbf{j} \in \tilde{\mathcal{E}}_n$ $f_{\mathbf{i}}(I) \cap f_{\mathbf{j}}(I) = \emptyset$. If we move forward to level $n + m$ then whatever happens in other elements of \mathcal{E}_n for each \mathbf{i} element of $\tilde{\mathcal{E}}_n$ we will have jointly independent processes $\{Y_n^{\mathbf{i}}\}_{n \in \mathbb{N}}$ (since what happens in different cylinders is independent of each other) with the same distribution as the original process $\{Y_n\}_n$. Since we choose disjoint level- n cylinders, we have that for any $m \geq 0$

$$Y_{n+m} \geq \sum_{i=1}^{\#\tilde{\mathcal{E}}_n} Y_m^i,$$

where Y_m^i are random variables that are independent of each other and also from Y_n and distributed according to Y_m . Since $\#\mathcal{E}_n > \lfloor \frac{Y_n}{3N} \rfloor$ Thus there exists a C such that

$$\mathbb{E}(Y_{n+m}) \geq \mathbb{E}(\#\tilde{\mathcal{E}}_n)\mathbb{E}(Y_m) \geq \frac{1}{C}\mathbb{E}(Y_n)\mathbb{E}(Y_m).$$

\square

The proof of the following Lemma is the first part of the proof of [4, Theorem 1], but for the convenience of the reader we repeat it.

Lemma 3.20.

$$(3.17) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log Y_n \leq \lambda \text{ a.s.}$$

Proof. From $\lambda = \sup_n \frac{1}{n} \log \mathbb{E}(Y_n)$ it follows that $\log \mathbb{E}(Y_n) \leq n\lambda$ for all n . Let $\eta > \lambda$. $\mathbb{P}(\log Y_n \geq n\eta) \leq \frac{\mathbb{E}(Y_n)}{e^{n\eta}} \leq e^{n(\lambda-\eta)}$. Since $e^{n(\lambda-\eta)}$ is summable for any $\eta > \lambda$ it follows from Borel-Cantelli lemma, that

$$\mathbb{P}(\limsup_{n \rightarrow \infty} \frac{1}{n} \log Y_n \leq \lambda) = 1.$$

□

Lemma 3.21. *For $P^*(\delta)$ defined in (3.14) it is true that:*

$$\overline{\dim}_B(\Lambda_{\mathcal{F}}(p)) \leq P^* \left(\frac{-\log p}{\log L} \right)$$

Proof. For $n \geq 0$ and $\mathbf{a} = (a_0, \dots, a_n) \in [N] \times [L]^n$, let

$$N_{\mathbf{a}} := \# \{ \mathbf{i} \in \mathcal{E}_n : f_{\mathbf{i}}(I) \cap J_{a_1 \dots a_n}^{a_0} \neq \emptyset \},$$

denote the number of level- n retained cylinders intersecting the interval $J_{a_1 \dots a_n}^{a_0}$. It is easy to see, that

$$Y_n = \# \{ \mathbf{a} \in [N] \times [L]^n : N_{\mathbf{a}} \geq 1 \}.$$

To upper bound λ we use the argument as in [11, first part of proof of Proposition]. We repeat this argument in this slightly different setting. For any $0 \leq s \leq 1$ it is true, by Jensen's inequality, that

$$\begin{aligned} \mathbb{E}(Y_n) &= \mathbb{E}(\# \{ \mathbf{a} \in [N] \times [L]^n : N_{\mathbf{a}} \geq 1 \}) \leq \mathbb{E} \left(\sum_{\mathbf{a} \in [N] \times [L]^n} N_{\mathbf{a}}^s \right) \\ &= \sum_{\mathbf{a} \in [N] \times [L]^n} \mathbb{E}(N_{\mathbf{a}}^s) \leq \sum_{\mathbf{a} \in [N] \times [L]^n} \mathbb{E}(N_{\mathbf{a}})^s \\ &= \sum_{\mathbf{a} \in [N] \times [L]^n} p^{sn} [e \cdot A_{a_1 \dots a_n} e_b]^s \leq N \sum_{a_1, \dots, a_n \in [L]^n} p^{sn} \|A_{a_1 \dots a_n}\|^s. \end{aligned}$$

Recall from 3.13 and 3.14

$$\begin{aligned} P(t) &= \frac{1}{\log L} \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\mathbf{a} \in [L]^n} \|A_{a_1, \dots, a_n}\|^t, \\ P^*(\delta) &= \inf_t \{ -\delta t + P(t) \}. \end{aligned}$$

Observe

$$\begin{aligned} \lambda &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(Y_n) \leq \inf_{0 \leq s \leq 1} \lim_{n \rightarrow \infty} \frac{1}{n} \log N \sum_{\mathbf{a} \in [L]^n} p^{sn} \|A_{a_1 \dots a_n}\|^s \\ &= \inf_{0 \leq s \leq 1} s \cdot \log(p) + \log(L) P(s) = \log(L) P^* \left(\frac{-\log p}{\log L} \right). \end{aligned}$$

By Lemma 3.20 and the definition of the upper box dimension the assertion follows. □

Proof of Theorem 1.12. It follows from Proposition 3.15, that there exists an $\alpha < \frac{\log M}{\log L} - 1$ such that $P^*(\alpha) < 1$. Choose p_0 such that $\frac{\log M}{\log L} - 1 > \alpha = \frac{-\log p_0}{\log L}$. It is clear that in this case $p_0 > \frac{L}{M}$, but since $P^*(\alpha) < 1$ by Lemma 3.21 the upper box dimension is smaller than 1. □

4. RANDOM MENGER SPONGE

4.1. Proof of Theorem 2.2.

4.1.1. *The dual nature of \mathcal{M}_p .* All until now we have always considered \mathcal{M}_p as an example of a coin tossing self-similar set. However, to prove Theorem 2.2, we need another interpretation of \mathcal{M}_p as a three-dimensional inhomogeneous Mandelbrot percolation set. The random set that we call inhomogeneous Mandelbrot percolation in this paper is simply named Mandelbrot percolation in [16], [18]. The motivation for this different interpretation of \mathcal{M}_p is that we would like to use the projection theorems of [18], which are stated for the inhomogeneous Mandelbrot percolation sets. To construct \mathcal{M}_p as an inhomogeneous Mandelbrot percolation set, we divide the unit cube $Q := [0, 1]^3$ into 27 axes parallel cubes of side length $1/3$.

$$\{\mathfrak{R}_{(i,j,k)}\}_{i,j,k=0}^2, \text{ where } \mathfrak{R}_{(i,j,k)} := \left[\frac{i}{3}, \frac{i}{3} + \frac{1}{3}\right] \times \left[\frac{j}{3}, \frac{j}{3} + \frac{1}{3}\right] \times \left[\frac{k}{3}, \frac{k}{3} + \frac{1}{3}\right].$$

Similarly, we define the level n cubes

$$\mathfrak{R}_n := \left\{ \mathfrak{R}_{(\mathbf{i}, \mathbf{j}, \mathbf{k})} := (\bar{\mathbf{i}}, \bar{\mathbf{j}}, \bar{\mathbf{k}}) + [0, 3^{-n}]^3 \right\}_{\mathbf{i}, \mathbf{j}, \mathbf{k} \in \{0, 1, 2\}^n},$$

where $\mathbf{i} := (i_1, \dots, i_n)$ and $\bar{\mathbf{i}} := \sum_{\ell=1}^n i_\ell 3^{-\ell}$

Let

$$\tilde{\mathcal{A}} := \{(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 1, 1)\}.$$

Note that these indices correspond to the cubes which are deleted when we construct the first approximation of the deterministic Menger sponge. For each $(i, j, k) \in \{0, 1, 2\}^3$ we set a probability $p_{(i,j,k)} \in [0, 1]$ as follows:

$$p_{(i,j,k)} := \begin{cases} 0, & \text{if } (i, j, k) \in \tilde{\mathcal{A}}; \\ p, & \text{otherwise.} \end{cases}$$

We retain the cube $\mathfrak{R}_{(i,j,k)}$ with probability $p_{(i,j,k)}$ independently. Assume that we have already constructed the level n retained cubes. That is we are given the random set $\mathcal{E}_n \subset \{0, 1, 2\}^n \times \{0, 1, 2\}^n \times \{0, 1, 2\}^n$ such that after n steps we have retained the cubes $\{\mathfrak{R}_{(\mathbf{i}, \mathbf{j}, \mathbf{k})}\}_{(\mathbf{i}, \mathbf{j}, \mathbf{k}) \in \mathcal{E}_n}$. For every $(\mathbf{i}, \mathbf{j}, \mathbf{k}) \in \mathcal{E}_n$ we retain the cube $\mathfrak{R}_{(i_{n+1}, j_{n+1}, k_{n+1})}$ with probability $p_{(i_{n+1}, j_{n+1}, k_{n+1})}$ independently of everything. Let $E_n := \bigcup_{(\mathbf{i}, \mathbf{j}, \mathbf{k}) \in \mathcal{E}_n} \mathfrak{R}_{(\mathbf{i}, \mathbf{j}, \mathbf{k})}$. Then

$$\mathcal{M}_p = \bigcap_{n=1}^{\infty} E_n.$$

Proposition 4.1. *For any plane $S(a, b, c)$:*

$$(4.1) \quad \frac{\mathcal{L}eb_2(Q \setminus M_1 \cap S(a, b, c))}{\mathcal{L}eb_2(Q \cap S(a, b, c))} \leq \frac{5}{9}.$$

where $\mathcal{L}eb_2$ denotes the two-dimensional Lebesgue measure, M_1 denotes the first approximation of the deterministic Menger sponge. Finally, $S(a, b, c)$ denote the plane

$$(4.2) \quad S(a, b, c) := \{(x, y, z) \in \mathbb{R}^3 : ax + by + c = z\}.$$

4.1.2. *proof of Proposition 4.1.* We use some of the results of Simon and Vágó [18] on projections of inhomogeneous Mandelbrot percolations. In [18] Simon and Vágó introduces conditions under which the projections of the inhomogeneous Mandelbrot percolation contains an interval almost surely conditioned on non-extinction. In what follows we show that for $p > 0.25$ these conditions are satisfied. Let $\Pi_{(a,b)}$ denote the projection along the planes $S(a, b, \cdot)$ to the z -coordinate axis, namely

$$\Pi_{(a,b)}(x, y, z) := z - ax - by.$$

In particular, $\Pi_{(a,b)}(S(a,b,t)) = t$. Observe that whenever $\tilde{c} \neq 0$ we have

$$\frac{1}{\tilde{c}} \text{proj}_{(\tilde{a}, \tilde{b}, \tilde{c})} \equiv \Pi_{(-\frac{\tilde{a}}{\tilde{c}}, -\frac{\tilde{b}}{\tilde{c}})}.$$

We remark that because of the symmetries of the Menger sponge, without loss of generality we can restrict our attention to those projections $\text{proj}_{\tilde{a}, \tilde{b}, \tilde{c}}$ for which $\tilde{c} \neq 0$. These projections can be identified with projections along planes of the form (4.2).

In particular, the following implication holds:

$$(4.3) \quad \text{int}(\Pi_{(a,b)}(\mathcal{M}_p)) \neq \emptyset, \quad \forall(a,b) \implies \text{int}(\text{proj}_{\tilde{a}, \tilde{b}, \tilde{c}}(\mathcal{M}_p)) \neq \emptyset, \quad \forall(\tilde{a}, \tilde{b}, \tilde{c}),$$

where int stands for the interior. Clearly, the inequality in (4.1) is equivalent to

$$(4.4) \quad \frac{\mathcal{L}eb_2(M_1 \cap S(a,b,c))}{\mathcal{L}eb_2(Q \cap S(a,b,c))} \geq \frac{4}{9}.$$

Proposition 4.2. *If the inequality in (4.4) holds for all (a,b,c) then almost surely, for all (a,b) , the projection $\Pi_{(a,b)}(\mathcal{M}_p)$ contains an interval for all $p > 0.25$.*

Proof. Suggested by the formula on the top of page 177 in [18] we consider the function

$$(4.5) \quad f(t) := \mathcal{L}eb_2(S(a,b,t) \cap Q).$$

A combination of [18, Theorem 1.2 and Proposition 2.3] applied to the projection $\Pi_{(a,b)}$ and function f yields the assertion of the Proposition. More precisely, if we replace proj_α in [18, Theorem 1.2] by $\Pi_{(a,b)}$ then we obtain that the assertion of Proposition 4.2 holds if a certain condition which is named as Condition $A(a,b)$ (see [18, Definition 2.1]) holds for all (a,b) . On the other hand, [18, Proposition 2.3] asserts in our case that another condition which is called Condition $B(a,b)$ (see [18, Definition 2.2]) implies that Condition $A(a,b)$ holds. It is immediate from the definitions that (4.4) implies that Condition $B(a,b)$ holds with f defined in (4.5) and projection $\Pi_{(a,b)}$. \square

Proof of Theorem 2.2 assuming Proposition 4.1. The assertion of Theorem 2.2 immediately follows from Proposition 4.2 and the implication in (4.3). \square

4.1.3. *Proof of Proposition 4.1.* We will frequently use the following simple observation.

Fact 4.3. *Observe that $Q \setminus M_1$ remains unchanged if we permute the coordinate axes and if we reflect to any of the planes $x = \frac{1}{2}$, $y = \frac{1}{2}$ or $z = \frac{1}{2}$.*

Hence, it is easy to see that without loss of generality we may assume

$$(4.6) \quad 0 \leq a \leq b \leq 1.$$

Under this assumption, the plane $S(a,b,c)$ intersects the unit cube $[0,1]^3$ if and only if

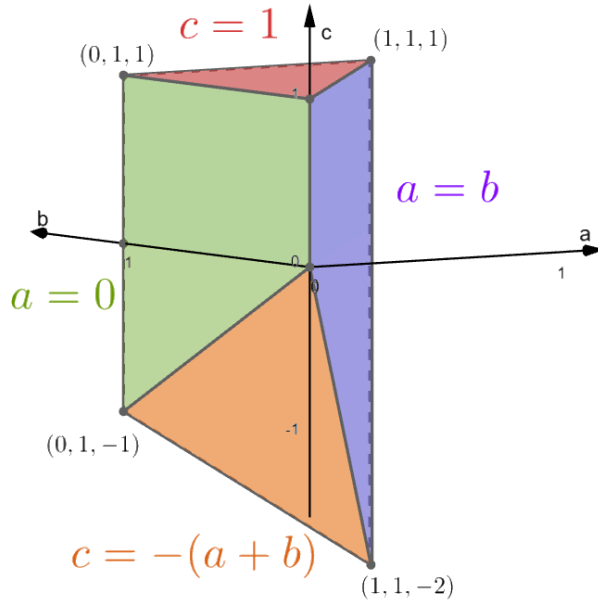
$$-(a+b) \leq c \leq 1.$$

So, we can confine ourselves to the region (see Figure 10)

$$(4.7) \quad \mathfrak{D} := \{(a,b,c) : 0 \leq a \leq b \leq 1, -(a+b) \leq c \leq 1\}.$$

We partition $\mathfrak{D} = \mathfrak{D}_{\text{CA}} \cup \mathfrak{D}_{\text{WM}}$, where

$$\mathfrak{D}_{\text{WM}} := \left\{ (a,b,c) \in \mathfrak{D} : a \in \left[\frac{1}{3}, 1 \right], b \in [a, 1], \right. \\ \left. c \in \left[\frac{2}{3} - (a+b), 0 \right] \cup \left[\max\{0, 1 - (a+b)\}, \frac{1}{3} \right] \right\},$$


 FIGURE 10. Illustration of the set \mathfrak{D} (4.7).

and

$$\mathfrak{D}_{CA} := \mathfrak{D} \setminus \mathfrak{D}_{WM}.$$

According to this partition, the proof is divided into two major parts. We verify (4.1)

- for $(a, b, c) \in \mathfrak{D}_{CA}$ with case analysis in Section 4.1.5.
- For $(a, b, c) \in \mathfrak{D}_{WM}$ we introduce a function $\tilde{H}(a, b, c)$ on \mathfrak{D} in (4.11), and we reformulate the inequality in (4.1) as $\tilde{H}|_{\mathfrak{D}_{WM}} \geq 0$. To verify this, in Section 4.1.6 we estimate the Lipschitz constant of $\tilde{H}|_{\mathfrak{D}_{WM}}$. In Section 4.1.7 we consider a sufficiently dense grid $G \subset \mathfrak{D}_{WM}$. Then using Wolfram Mathematica we calculate $C := \min \tilde{H}|_G$. We verify that $C > 0$ is sufficiently large that because of the Lipschitz property of \tilde{H} , we get that $\tilde{H}|_{\mathfrak{D}_{WM}} > 0$.

First we introduce the notation used in the rest of Section 4.1.3.

4.1.4. *Notations.* We write $\text{proj}_{x,y}$ for the orthogonal projection to the (x, y) -coordinate plane, that is

$$\text{proj}_{x,y} : \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad \text{proj}_{x,y}((u, v, w)) := (u, v).$$

The area of the unit cube intersected with the plane $S(a, b, c)$ and the area of the $\text{proj}_{x,y}$ -projection are respectively denoted by

$$F(a, b, c) := \mathcal{L}eb_2(Q \cap S(a, b, c)), \quad \tilde{F}(a, b, c) := \mathcal{L}eb_2(\text{proj}_{x,y}(Q \cap S(a, b, c))).$$

It is easy to see that

$$(4.8) \quad \tilde{F}(a, b, c) \sqrt{1 + a^2 + b^2} = F(a, b, c).$$

Let

$$\mathcal{A} := \left\{ \left(0, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, 0, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, 0\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right) \right\}$$

denote the lower left corners (the closest point of the cube to the origin) of the level-1 cubes missing from the first approximation of the Menger sponge. Also when we want to calculate the area of a level 1 square with lower left corner (u, v, w) intersected with the plane $S(a, b, c)$ we renormalize the plane with respect to the cube and calculate the area of the renormalized intersection and divide it by 9 (see (4.10)). Renormalization means that we transform a level 1 cube into the unit cube. In case of a level one cube $(u, v, w) + [0, \frac{1}{3}]^3$ this transformation is:

$$(x, y, z) \rightarrow 3(x - u, y - v, z - w).$$

The renormalization of the plane $S(a, b, c)$ with respect to the level 1 cube with lower left corner (u, v, w) is $S(a, b, c')$, where $c' := 3(au + bv + c - w)$. This is why we introduce

$$(4.9) \quad g(a, b, c, u, v, w) := (a, b, 3(au + bv + c - w)).$$

Observe that for a level 1 cube $\widehat{Q} := (u, v, w) + [0, \frac{1}{3}]^3$:

$$(4.10) \quad \mathcal{L}eb_2(\widehat{Q} \cap S(a, b, c)) = \frac{1}{9}F(g(a, b, c, u, v, w)).$$

Hence, to verify Proposition 4.1 we need to show that for any (a, b, c) :

$$\begin{aligned} H(a, b, c) &:= \frac{5}{9}\mathcal{L}eb_2(Q \cap S(a, b, c)) - \mathcal{L}eb_2(Q \setminus M_1 \cap S(a, b, c)) \\ &= \frac{5}{9}F(a, b, c) - \frac{1}{9} \sum_{(u,v,w) \in \mathcal{A}} F(g(a, b, c, u, v, w)) \geq 0. \end{aligned}$$

Let

$$(4.11) \quad \widetilde{H}(a, b, c) := \frac{5}{9}\widetilde{F}(a, b, c) - \frac{1}{9} \sum_{(u,v,w) \in \mathcal{A}} \widetilde{F}(g(a, b, c, u, v, w)).$$

By (4.8) $H(a, b, c) = \frac{1}{\sqrt{1+a^2+b^2}}\widetilde{H}(a, b, c)$. In this way we have proved that

Fact 4.4. *The inequality in (4.1) holds if*

$$(4.12) \quad \widetilde{H}(a, b, c) \geq 0, \quad \text{for any } 0 \leq a \leq b \leq 1 \text{ and any } c.$$

Let also

$$(4.13) \quad \ell_i(x) := -\frac{a}{b}x + \frac{\frac{i}{3} - c}{b}, \quad i \in \{0, 1, 2, 3\}$$

denote the line which is the $\text{proj}_{x,y}$ -projection of

$$\left\{ \left(x, y, \frac{i}{3} \right) : x \in \mathbb{R}, y \in \mathbb{R} \right\} \cap S(a, b, c).$$

That is $y = \ell_i(x)$ is the $\frac{i}{3}$ -level set of $S(a, b, c)$. Note that (4.6) implies that the slope of $\ell_i(x)$ is between -1 and 0 .

For any set $H \subseteq [0, 1]^2$ the **bad part of H** is $H \cap \text{proj}_{x,y}(S(a, b, c) \cap Q \setminus M_1)$ and the **good part of H** is $H \cap \text{proj}_{x,y}(S(a, b, c) \cap M_1)$. The bad part of $[0, 1]^2$ is denoted by dark color on the figures below. Let $Q_{i,j} := (\frac{i}{3}, \frac{j}{3}) + [0, \frac{1}{3}]^2$, $i, j \in \{0, 1, 2\}$ as shown in the Figure 11A. We call these squares **level-1 squares**. By the construction of the Menger sponge, the squares in the corners can not have bad parts. The bad parts of $Q_{1,1}$ is between the lines $\ell_0(x)$ and $\ell_3(x)$, and the bad parts of $Q_{1,0}, Q_{2,1}, Q_{0,1}, Q_{1,2}$ are between the lines $\ell_1(x)$ and $\ell_2(x)$.

First we show by a detailed case analysis (Section 4.1.5) that for some values of a, b and c the assertion (4.12) holds. Then for the uncovered values of a, b, c we create a grid to

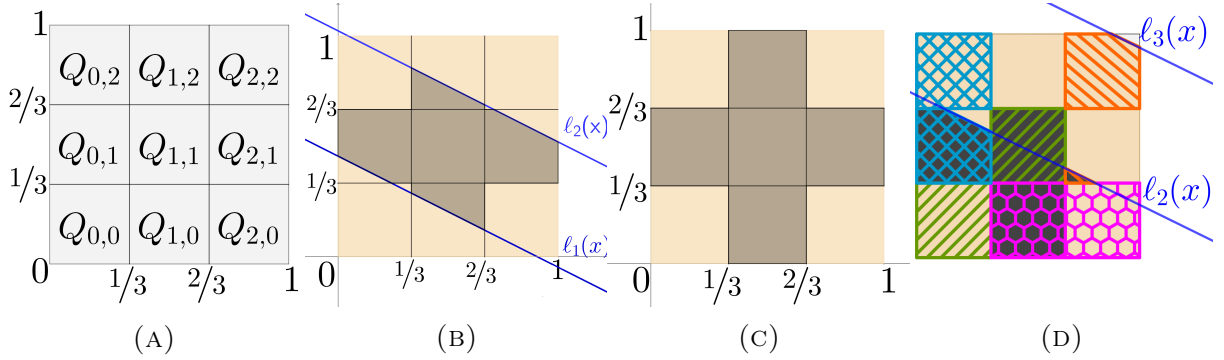


FIGURE 11. Figures for Fact 4.5 and 4.6. In the last three figures the lighter color denote the good and the darker the bad parts of the unit square respectively.

calculate the value of \tilde{H} on the grid points and estimate the value of the function out of the grid points, using the upper bound (Section 4.1.6) on the Lipschitz constant for the function \tilde{H} .

4.1.5. *Case analysis.* Recall that we always assume that $0 \leq a \leq b \leq 1$. Note that if $a = b = 0$ and $c \in (\frac{1}{3}, \frac{2}{3})$ then $\tilde{H}(a, b, c) = 0$.

Fact 4.5. *If $c > 0$ and $a + b + c \leq 1$, then $\tilde{H}(a, b, c) \geq 0$.*

Proof. Whenever $c > 0$ and $a + b + c \leq 1$ by elementary geometry we have $\tilde{F}(a, b, c) = 1$. Hence, $\tilde{H}(a, b, c)$ is the smallest when all of $Q_{1,0}, Q_{2,1}, Q_{0,1}, Q_{1,2}$ are between the lines $\ell_1(x)$ and $\ell_2(x)$. In this case, $\tilde{H}(a, b, c) = 0$. For visual explanation see Figures 11B and 11C. \square

Fact 4.6. *If $\frac{1}{3} \leq c \leq 1$, then $\tilde{H}(a, b, c) \geq 0$.*

Proof. Whenever $a + b + c \leq 1$ we are done by Fact 4.5. Hence, we can assume that $a + b + c > 1$. In this case $\ell_0([0, 1]), \ell_1([0, 1]) \leq 0$. This follows from $c \geq \frac{1}{3}$ and (4.13). For any such $S(a, b, c)$ plane if we fix the line $\ell_3(x)$ it is easy to see that the worst case scenario occurs when $c = \frac{1}{3}$, since for a fixed $\ell_3(x)$ the function $\tilde{F}(a, b, c)$ remains the same but the area of the bad part grows as we decrease c . Hence we may assume that $c = \frac{1}{3}$. We know that $\ell_3(1) < 1$ (because $a + b + c > 1$), so $a + b > \frac{2}{3}$. So, it completes the proof of Fact 4.6 if we verify the following Claim:

Claim 4.7. *If $a + b + c > 1$ and $c = \frac{1}{3}$ then for every level-1 square $Q_{i,j}$ we can find another level-1 square $Q_{i',j'}$ such that the area of the good part of $Q_{i',j'}$ is greater than or equal to the area of the bad part of $Q_{i,j}$. Moreover, distinct (i', j') correspond to distinct (i, j) .*

Namely, this claim implies that the area of the bad part of the unit square is smaller than the area of its good part and hence, $\tilde{H}(a, b, c) \geq 0$.

The proof of Claim 4.7 will be given below, but first we remark that this proof is illustrated on Figure 11D. The light background regions on Figure 11D refer to the good parts and the dark background regions refer to the bad parts. Figure 11D indicates that the bad region of every level-1 square is compensated by an identically patterned good region of a corresponding level 1 square having at least as large area as the corresponding bad region.

Proof of Claim 4.7.

- $Q_{1,2}$ does not have a bad part in this case, because $a + b > \frac{2}{3}$ and $b > \frac{1}{3}$, hence $a + 2b > 1$ thus, $\ell_2(\frac{1}{3}) = \frac{1-a}{3b} < \frac{2}{3}$. All of the lines $\ell_i(x)$ have negative slope, hence this shows that all of the points of $\ell_2(x)$ are under the square $Q_{1,2}$.
- The bad part of $Q_{1,1}$ is compensated by the good part of $Q_{0,0}$, since $\ell_3(0) \geq \ell_3(\frac{1}{3}) - \frac{1}{3}$ and hence, the line segment $\ell_3([0, \frac{1}{3}])$ is higher relative to the line $y = 0$ than the segment $\ell_3([\frac{1}{3}, \frac{2}{3}])$ to the line $y = \frac{1}{3}$. This part is illustrated by green lines on Figure 11D.
- The area of the bad part of $Q_{1,0}$ is smaller than the area of the good part of $Q_{2,0}$ since $\ell_3(\frac{2}{3}) \geq \ell_2(\frac{1}{3})$ and $\ell_2(x)$ and $\ell_3(x)$ have the same slope. This part is illustrated by pink honeycombs on Figure 11D.
- The area of the bad part of $Q_{2,1}$ is at most as much as the area of the good part of $Q_{2,2}$. This follows from the fact that $\ell_3(\frac{2}{3}) - \frac{2}{3} \geq \ell_2(\frac{1}{3}) - \frac{1}{3}$. This part is illustrated by orange lines on Figure 11D.
- The area of the bad part of $Q_{0,1}$ is at most as much as the area of the good part of $Q_{0,2}$. This follows from the fact that $\ell_3(0) - \frac{2}{3} \geq \ell_2(0) - \frac{1}{3}$. This part is illustrated by blue lines (crossed) on Figure 11D.

□

As we mentioned above, Claim 4.7 implies that the assertion of Fact 4.6 holds. □

Fact 4.8. *If $a + b + c \leq \frac{2}{3}$, then $\tilde{H}(a, b, c) \geq 0$.*

Proof. This is immediate from Facts 4.3 and 4.6. □

Corollary 4.9. *If $a + b \leq \frac{2}{3}$, then $\tilde{H}(a, b, c) \geq 0$.*

Proof. Whenever

$0 < c \leq \frac{1}{3}$: we are done by Fact 4.5;

$\frac{1}{3} \leq c \leq 1$: $a + b < 1 - c$: we are done by Fact 4.5,

$a + b \geq 1 - c$: we are done by Fact 4.6;

$c \leq 0$: we are done by Fact 4.8.

□

Fact 4.10. *If $a < \frac{1}{3}$, then $\tilde{H}(a, b, c) \geq 0$.*

Proof. Let $h_i := \ell_i(0)$ and $\hat{h}_i := \ell_i(1)$. Since $\text{dist}(h_i, h_{i+1}) = \frac{1}{3b}$ and $b \leq 1$ it follows that $\text{dist}(h_i, h_{i+1}) \geq \frac{1}{3}$, hence the following cases are possible:

- I. $h_0 \in [0, \frac{1}{3}]$
 - I.a. $h_1 \in [\frac{1}{3}, \frac{2}{3})$
 - I.a.1. $h_2 \in [\frac{2}{3}, 1)$
 - I.a.2. $h_2 \geq 1$
 - I.b. $h_1 \in [\frac{2}{3}, 1)$ and $h_2 \geq 1$
 - I.c. $h_1 \geq 1$ and $h_2 > 1$
- II. $h_0 \in [\frac{1}{3}, \frac{2}{3})$
 - II.a. $h_1 \in [\frac{2}{3}, 1)$ and $h_2 \geq 1$
 - II.b. $h_1 \geq 1$ and $h_2 > 1$
- III. $h_0 \in [\frac{2}{3}, 1)$ and $h_1 \geq 1$ and $h_2 > 1$
- IV. $h_0 \geq 1$
- V. $h_0 < 0$.

The proof of Case V. follows from Fact 4.3, Fact 4.5 and from Cases I-IV. We mainly use two methods for the proof.

One is to estimate the area in the case when $a = 0$, i.e. the slopes of the lines ℓ_i are zero and to upper bound the growth of the area of the bad part and lower bound the shrinkage of the area of the good part to get a "worst case scenario" fraction of the bad and the total area. We use this method in case of I.a.1.-I.b..

The other way is to prove that the good part of the unit square is at least as big as the bad part. We do this one-by-one for each $Q_{i,j}$ that has a bad part. We apply this method in case of I.c.-IV..

Case I.a.1.: In this case $h_0 \in [0, \frac{1}{3}]$, $h_1 \in [\frac{1}{3}, \frac{2}{3}]$, $h_2 \in [\frac{2}{3}, 1)$ and $h_3 \geq 1$. We distinguish two cases: when $a = 0$ or $a > 0$.

$a = 0$: Then $\ell_i(x) = h_i$ and $\tilde{F}(a, b, c) = 1 - h_0$. $Q_{1,1}$ has only bad part since $h_0 \leq \frac{1}{3}$ and $h_1 \geq 1$, $Q_{1,0}$ has no bad parts because $h_1 \geq \frac{1}{3}$. The area of the bad part of $Q_{0,1}$ and $Q_{2,1}$ together is $\frac{2}{3}(\frac{2}{3} - h_1)$ and the area of the bad part of $Q_{1,2}$ is $\frac{1}{3}(h_2 - \frac{2}{3})$. So, the bad area in $[0, 1]^2$ is $1/3$ of the total area which is $\tilde{F}(a, b, c) = 1 - h_0$.

$a > 0$: In this case we distinguish two further sub-cases. Namely, the first is when $\ell_3(1) \geq 1$ (see Figure 12A) and the second one is when $\ell_3(1) < 1$ (see Figure 12B). In both sub-cases the area of the bad part increases at most with the area of the triangle $(0, h_1), (1, h_1), (1, \hat{h}_1)$ intersected with $Q_{0,1}$ and $Q_{1,0}$ and $Q_{2,1}$ (see the light hatched area on Figure 12B). This area is smaller than $\frac{a}{3b}$.

$\ell_3(1) \geq 1$: Then the total area (the area of the good and the bad part together) does not decrease. Rewriting $h_2 = h_0 + \frac{2}{3b}$ and $h_1 = h_0 + \frac{1}{3b}$ gives that $\tilde{H}(a, b, c) \geq \frac{2}{9}(1 - h_0) - \frac{a}{3b}$. It follows from $h_2 = \frac{2-c}{b} < 1$, that $b + c > \frac{2}{3}$ and from $\frac{2}{3b} = h_2 - h_0 \leq 1$, that $b > \frac{2}{3}$. Multiplying $\frac{2}{9}(1 - h_0) - \frac{a}{3b}$ by $9b > 0$ gives $2b + c - 3a > \frac{1}{3} > 0$. Thus, $\tilde{H}(a, b, c) > 0$.

$\ell_3(1) < 1$: Then the total area decreases. An easy calculation shows that the total area is $1 - \frac{c^2}{2ab} - \frac{(a+b+c-1)^2}{2ab}$. Observe that $\ell_3(1) = -\frac{a}{b} + \frac{1-c}{b} < 1$ implies that $a + b + c > 1$. Using this and that $b > \frac{2}{3}$ a substantial but elementary calculation shows that in this case $\tilde{H}(a, b, c) > 0$.

Case I.a.2.: In this case $\hat{h}_3 > 1$ since $\hat{h}_3 > h_2 > 1$. The proof is very similar to the one of I.a.1..

$a = 0$: In this case the total area $\tilde{F}(a, b, c)$ is $1 - h_0$. From $h_1 \geq \frac{1}{3}$ it follows that $Q_{1,0}$ does not have a bad part. Hence, the bad parts and their areas are as follows:

- $Q_{1,1}$ and $Q_{1,2}$ consists only of bad parts, hence their bad area together is $\frac{2}{9}$,
- $Q_{0,1}$ and $Q_{2,1}$ have bad parts over h_1 , the area of their bad parts together is $(\frac{2}{3} - h_1) \cdot \frac{2}{3}$

Altogether the area of the bad part is

$$\frac{2}{3}(1 - h_0) - \frac{2}{9b}.$$

$2b(1 - h_0) < 2$, hence $6b(1 - h_0) < 2 + 4b(1 - h_0)$ thus, $\frac{2}{3} - \frac{2}{9b(1-h_0)} < \frac{4}{9}$, hence the fraction of the bad area and the total is smaller than $\frac{4}{9} < \frac{5}{9}$.

$a > 0$: When $a > 0$, it is clear that the area of the bad part grows with at most $\frac{a}{3b}$ when we increase a from 0, similarly to the previous case. In contrast the total area does not decrease, hence we can use the previous calculation to show that in this case the assertion holds.

Case I.b.:

$a = 0$: The total area is again $(1 - h_0)$. It follows from $h_1 > \frac{2}{3}$, that only $Q_{1,2}$ and $Q_{1,1}$ can have bad parts. It is easy to see, that $Q_{1,1}$ has only bad parts, and that the bad part of $Q_{1,2}$ has area $(1 - h_1)\frac{1}{3} = \frac{1}{3}(1 - h_0 - \frac{1}{3b})$. An easy calculation shows that in this case the fraction of the bad and the total area is smaller than $\frac{4}{9}$, hence we are done.

$a > 0$: Similarly to the earlier cases we consider the growth of the area if we increase a from 0. The additional bad part is the intersection of $Q_{0,1} \cup Q_{2,1} \cup Q_{1,2}$ with the triangle which we get by intersecting the unit square with the area between ℓ_1 and the line with height h_1 . The area of this triangle is smaller than $\frac{a}{3b}$. The assertion follows if we rewrite the inequality $\frac{1}{9(1-h_0)} + \frac{1}{3} + \frac{3a-1}{9b(1-h_0)} < \frac{5}{9}$, getting $1 + \frac{3a-1}{b} < 2(1 - h_0)$, which trivially holds, since $3a - 1 < 0$, hence the left-hand side is smaller than 1, and the right hand side is greater than 1 by $1 - h_0 > \frac{2}{3}$

Case I.c.: This case is almost identical to II.b., only in that case the total area might be smaller because h_0 is greater. For this reason, instead of proving this statement we present the proof for Case II.b. later because that might seem more complicated.

Case II.a. Observe that from $\ell_1(1) \geq h_0 \geq \frac{1}{3}$ it follows that $Q_{1,0}$ does not have bad part.

$Q_{1,2}$: The bad parts of $Q_{1,2}$ are compensated by the good parts of $Q_{0,2}$. This is because $Q_{0,2}$ only has good parts, because $\ell_3(x) \cap [0, 1] > 1$ and $\ell_0(0) \leq \frac{2}{3}$.

$Q_{1,1}$: $Q_{2,2}$ only has good parts, which can be verified in the same way as in the case of $Q_{0,2}$. Thus, the bad part of $Q_{1,1}$ is compensated by $Q_{2,2}$.

$Q_{0,1}$: Since $b \leq 1$, $\ell_1(x) - \ell_0(x) = \frac{1}{3b} \geq \frac{1}{3}$. Consequently, the area of $Q_{0,0}$ above the line $\ell_0(x)$ (i.e. the good part of $Q_{0,0}$) is greater than the area of $Q_{0,1}$ above $\ell_1(x)$ (i.e. the bad part of $Q_{0,1}$). It follows, that the area of the bad part of $Q_{0,1}$ is smaller than the area of the good part of $Q_{0,0}$.

$Q_{2,1}$: By almost exactly the same reasoning as in the previous case, the area of the bad part of $Q_{2,1}$ is smaller than the good part of $Q_{2,1}$.

Case II.b.: In this case $h_0 \in [\frac{1}{3}, \frac{2}{3})$, $h_1 \geq 1, h_2, h_3 > 1$. Note that $a < \frac{1}{3}$ yields $\ell_3([0, 1]) > h_1 \geq 1$ and since $h_0 < \frac{2}{3}$ the squares $Q_{0,1}$ and $Q_{2,2}$ have only good parts. It follows from $a < \frac{1}{3}$ that $\ell_1([0, 1]) > h_0 \geq \frac{1}{3}$ and $h_1 \geq 1$. Potentially only $Q_{1,1}, Q_{1,2}$ and $Q_{2,1}$ have bad parts. For the illustration of the following pairing see Figure 12c, where the bad parts are denoted by the darker color, the good parts are denoted by the lighter color and the pairs have the same pattern and color.

$Q_{1,1}$: Since the area of the bad part of $Q_{1,1}$ is at most $\frac{1}{9}$, and the area of $Q_{2,2}$ is $\frac{1}{9}$ the area of the bad part of $Q_{1,1}$ is compensated by the area of the good part of $Q_{2,2}$.

$Q_{1,2}$: The same holds for the bad area of $Q_{1,2}$ and the good area of $Q_{0,2}$.

$Q_{2,1}$: $Q_{2,1}$ has bad part only in the case when $\widehat{h}_1 < \frac{2}{3}$. The good part of $Q_{2,0}$ which is above $\ell_0(x)$ and the bad part of $Q_{2,1}$ which is above $\ell_1(x)$ and since $\ell_1(x)$ and $\ell_0(x)$ has the same slope, it is enough to show that $\frac{1}{3} - \widehat{h}_0 > \frac{2}{3} - \widehat{h}_1$. Since $\widehat{h}_1 = \widehat{h}_0 + \frac{1}{3b}$ the inequality reduces to $1 > b$, which is true by our assumptions, hence the area of the good part of $Q_{2,0}$ is greater than the area of the bad part of $Q_{2,1}$.

Case III. It is clear that only $Q_{1,1}$ and $Q_{1,2}$ can have bad parts, since $\ell_1 \cap [0, 1] \geq h_0 \geq \frac{2}{3}$.

$Q_{1,1}$: Whenever $Q_{1,1}$ has bad parts, $\ell_0(\frac{2}{3}) \leq \frac{2}{3}$, then the whole $Q_{2,2}$ is good. Thus, it compensates for the bad parts of $Q_{1,1}$.

$Q_{1,2}$: Since $\frac{1-a}{3b} > 0$ it is easy to see that the area of the good part of $Q_{0,2}$ is greater than the area of the bad part of $Q_{1,2}$.

Case IV. In this case only $Q_{1,1}$ can have bad part, and if it happens then the whole $Q_{2,2}$ is good, hence the bad area of $Q_{1,1}$ is compensated. \square

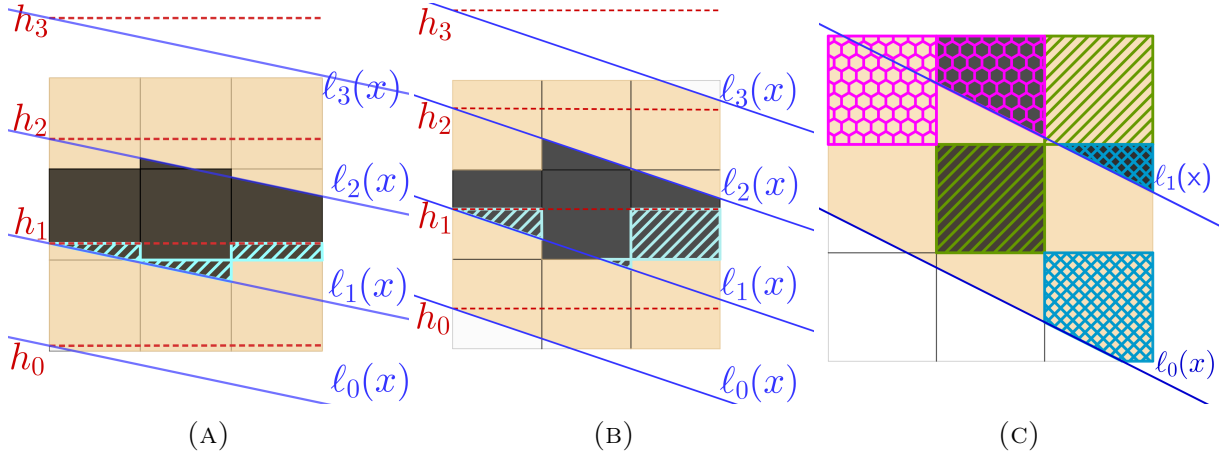


FIGURE 12. Figures for Fact 4.10. In the first two figures the red lines denote the height of h_i , the blue lines denote $\ell_i(x)$.

4.1.6. *Lipschitz-constants for the function $\tilde{F}(a, b, c)$.* Putting together Facts 4.10 and 4.4 from now on we may always assume that

$$(4.14) \quad \frac{1}{3} \leq a \leq b \leq 1.$$

Since the gradient of the renormalized plane remains the same these assumptions hold when instead of \tilde{F} we consider $\tilde{F} \circ g$ (g was defined in (4.9)). For a fix (a, b) satisfying (4.14) consider the plane $S(a, b, c)$. This plane can intersect the unit cube $[0, 1]^3$ in eight different ways depending on c . The different ways of intersections yields different formulas for $\tilde{F}(a, b, c)$.

We estimate the Lipschitz constant from above by the sup-norm of the gradient of \tilde{F} , which can be calculated using elementary calculus. Table 1 contains the results of the calculation, and some figures to visualize the different cases.

$$a + b < 1$$



$$a + b > 1$$

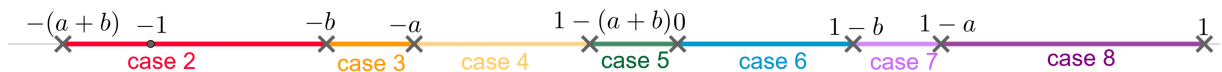


FIGURE 13. Figure for Table 1. The two figures show that for a fixed a and b what case does the different values of c give. The cases are named after the first column of Table 1, e.g. when $-b \leq c \leq -a$ we are in case 3. In the upper figure we consider the case, when $a + b < 1$, in the lower, when $a + b > 1$.



















Name	Figure1	Figure2	Conditions	$\tilde{F}(a, b, c)$	Lipsch.
0			$c \geq 1$ or $a + b + c \leq 0$	0	0
1			$c \geq 0$ and $a + b + c \leq 1$	1	0
2			$-(a + b) \leq c \leq b$	$\frac{(a+b+c)^2}{2ab}$	$\frac{3\sqrt{3}}{2}$
3			$-b \leq c \leq -a$	$\frac{a+2b+2c}{2b}$	$\frac{3\sqrt{3}}{2}$
4			$-a \leq c$ and $c \leq \min \{0, 1 - (a + b)\}$	$1 - \frac{c^2}{2ab}$	$\frac{3\sqrt{3}}{2}$
5			$1 \leq a + b$ and $1 - (a + b) \leq c \leq 0$	$1 - \frac{c^2 + (a+b+c-1)^2}{2ab}$	$\frac{131}{54}$
6			$\max \{0, 1 - (a + b)\} \leq c$ and $c \leq 1 - b$	$1 - \frac{(a+b+c-1)^2}{2ab}$	$\frac{3\sqrt{3}}{2}$
7			$1 - b \leq c \leq 1 - a$	$\frac{2-2c-a}{2b}$	$\frac{3\sqrt{3}}{2}$
8			$1 - a \leq c \leq 1$	$\frac{(1-c)^2}{2ab}$	$\frac{3\sqrt{3}}{2}$

TABLE 1. The first column contains the name (used in Figure 13) of the case. The second column shows the way of the intersection of the plane and the unit cube, the third column shows the projection of the above. In the fourth – conditions – column, we describe the region of (a, b, c) under which we are in the given case. The fifth column contains the value of the function \tilde{F} in the given case, and the last, Lipsch., column contains the upper bound for the Lipschitz constant in the given case.

Since according to Table 1 all the Lipschitz-constants are less than 4,

$$(4.15) \quad \left\| \tilde{F}(a, b, c) - \tilde{F}(\hat{a}, \hat{b}, \hat{c}) \right\| < 4 \left\| (a, b, c) - (\hat{a}, \hat{b}, \hat{c}) \right\|.$$

Using that

$$A := \nabla_{a,b,c} g(a, b, c, u, v, w) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3u & 3v & 3 \end{bmatrix}$$

and that $\sup_{(u,v,w) \in \mathcal{A}} \|A\| < 4$, it follows that

$$\begin{aligned}
 (4.16) \quad & \left\| \sum_{(u,v,w) \in \mathcal{A}} \tilde{F}(g(a, b, c, u, v, w)) - \tilde{F}(g(\hat{a}, \hat{b}, \hat{c}, u, v, w)) \right\| \\
 & < \sum_{(u,v,w) \in \mathcal{A}} 4 \cdot \left\| g(a, b, c, u, v, w) - g(\hat{a}, \hat{b}, \hat{c}, u, v, w) \right\| \leq |\mathcal{A}| \cdot 4 \cdot 4 \left\| (a, b, c) - (\hat{a}, \hat{b}, \hat{c}) \right\| \\
 & \leq 7 \cdot 4 \cdot 4 \cdot \left\| (a, b, c) - (\hat{a}, \hat{b}, \hat{c}) \right\|. \\
 & \left\| \tilde{H}(a, b, c) - \tilde{H}(\hat{a}, \hat{b}, \hat{c}) \right\| \leq \frac{5}{9} \left\| \tilde{F}(a, b, c) - \tilde{F}(\hat{a}, \hat{b}, \hat{c}) \right\| \\
 & + \frac{1}{9} \left\| \sum_{(u,v,w) \in \mathcal{A}} \tilde{F}(g(a, b, c, u, v, w)) - \tilde{F}(g(\hat{a}, \hat{b}, \hat{c}, u, v, w)) \right\| < \frac{44}{3} \left\| (a, b, c) - (\hat{a}, \hat{b}, \hat{c}) \right\|,
 \end{aligned}$$

where the second inequality follows from the combination of (4.15) and (4.16). Hence,

$$(4.17) \quad \tilde{H}(\hat{a}, \hat{b}, \hat{c}) > \tilde{H}(a, b, c) - 15 \left\| (a, b, c) - (\hat{a}, \hat{b}, \hat{c}) \right\|.$$

4.1.7. Numerical calculations using Wolfram Mathematica. First we construct a grid G depending on a given positive number \hat{d} . Let $a_0 = \frac{1}{3}$ and if a_n is defined and $a_n + \hat{d} \leq 1$, let $a_{n+1} = a_n + \hat{d}$, otherwise $I = n$. For a given $i \in \{0, \dots, I\}$, let $b_{i,0} = a_i$ and if $b_{i,n} + \hat{d} \leq 1$ then we define $b_{i,n+1} = b_{i,n} + \hat{d}$, otherwise $J_i = n$. For a given $i \in \{0, \dots, I\}$, $j \in \{0, \dots, J_i\}$, let $c_{i,j,0} = \frac{2}{3} - (a_i + b_j) \leq 0$ and if $c_{i,j,n}$ is defined and $c_{i,j,n} + \hat{d} \leq \frac{1}{3}$ then we put $c_{i,j,n+1} = c_{i,j,n} + \hat{d}$ otherwise $K_{i,j} = n$. Now if we choose a point $(\hat{a}, \hat{b}, \hat{c})$ such that $\frac{1}{3} \leq \hat{a} \leq 1$, $\hat{a} \leq \hat{b} \leq 1$ and $\frac{2}{3} - (\hat{a} + \hat{b}) \leq \hat{c} \leq \frac{1}{3}$, then there exists a grid point $(a_i, b_{i,j}, c_{i,j,k}) \in G$, $i \in \{0 \dots I\}$, $j \in \{0, \dots, J_i\}$, $k \in \{0, \dots, K_{i,j}\}$ such that $\|(\hat{a}, \hat{b}, \hat{c}) - (a_i, b_{i,j}, c_{i,j,k})\| < \sqrt{3}\hat{d}$. Observe that (4.17), yields the following implication: If $\tilde{H}(a, b, c) \geq 15 \cdot \sqrt{3} \cdot \hat{d}$ holds for every $(a, b, c) \in G$ then we have $\tilde{H}(a, b, c) \geq 0$ for every (a, b, c) . We choose $\hat{d} = \frac{1}{500}$ and with Wolfram Mathematica we evaluate \tilde{H} at every grid point. We obtain that the minimum of the results is $\frac{62509}{1125000}$. This is greater than $15 \cdot \sqrt{3} \cdot \frac{1}{500}$. This finishes the proof of Proposition 4.1.

5. APPENDIX

Here we present the proof of Proposition 3.15. This is a simple consequence of a combination of a number of theorems due to Ruiz [17] and Bárány and Rams [3].

Proof of Proposition 3.15. Throughout the Appendix we use the notations of Section 1.2. The proof is divided into one Fact and five Lemmas.

Fact 5.1. For any $\ell \in [N]$, $b \in [L]$ and $\mathbf{a} \in [L]^n$, $n \geq 0$,

$$\nu(J_{b\mathbf{a}}^\ell) = \sum_{k \in [N]} \frac{1}{M} A_b(\ell, k) \nu(J_{\mathbf{a}}^k).$$

This fact can be proved by the same argument used in [17, bottom of page 354].

For an $\mathbf{a} \in [L]^n$ we define the vectors:

$$\boldsymbol{\nu}(\cdot, \mathbf{a}) = (\nu(J_{\mathbf{a}}^0), \dots, \nu(J_{\mathbf{a}}^{N-1})), \quad \boldsymbol{\nu}(\cdot, \emptyset) = (\nu(J^0), \dots, \nu(J^{N-1})).$$

Fact 5.1 implies that $\nu(J_{\mathbf{a}}^\ell) = \frac{1}{M^n} \mathbf{e}_\ell^T \cdot A_{\mathbf{a}} \boldsymbol{\nu}(\cdot, \emptyset)$, where $\mathbf{e}_\ell \in \mathbb{R}^N$ is the ℓ -th coordinate unit vector. Let

$$(5.1) \quad \eta_j := \nu_j \circ \mathbf{t}_j, \quad \eta := \sum_{j \in [N]} \eta_j$$

where $\nu_j = \nu_{|J_j}$.

It is easy to see that η is probability measure on $[0, L]$. We also define

$$\tilde{\eta}([a_1, \dots, a_n]) := \eta(J_{a_1, \dots, a_n}^0).$$

Observe that

$$\eta(J_{\mathbf{a}}^0) = \sum_{j \in [N]} \nu(J_{\mathbf{a}}^j) = \frac{1}{M^n} \mathbf{e}^T A_{\mathbf{a}} \boldsymbol{\nu}(\cdot, \emptyset).$$

Now we briefly explain the intuition behind the method presented below. Instead of studying the original attractor we consider the one which we get by intersecting the original attractor with the different intervals J^0, \dots, J^N , and translate all of the resulting intersection sets to $[0, L]$. The measure η can be thought of as the natural measure of the modified system corresponding to the modified attractor. Also to be able to connect the matrices A_a and their products to the behaviour of the system it is more convenient for us to work in the symbolic space $\Sigma^{(L)}$ containing the L -adic codes of the points of $[0, L]$ (the support of our new attractor).

Lemma 5.2. *The matrix $A = \sum_{k \in [N]} A_k$ is primitive, meaning that there exists a $K > 0$ such that $A^K(i, j) > 0$ for every $i, j \in \{1, \dots, N\}$.*

Proof. It is proven in [2, Section 4.4], but for the convenience of the reader we present the proof of this lemma. Recall that for any $\ell \in [N]$, $\nu(J^\ell) > 0$. It follows from [17] that for any ℓ there exists an x_ℓ such that $x_\ell \in \text{Int}(J^\ell) \cap \Lambda$. Since $x_\ell \in \Lambda$, there exists an $\mathbf{i}^\ell \in \Sigma^{(M)}$ such that $x_\ell = \Pi^{(M)}(\mathbf{i}^\ell)$.

Recall that $|I| = \tilde{n}L$. Let d_ℓ be the distance between x_ℓ and the nearest end-point of J_ℓ , let $d' > 0$ be the minimum of the numbers d_ℓ . We choose K such that $|f_{\mathbf{i}^\ell|_K}(I)| = \frac{\tilde{n}}{L^{K-1}} < d'$. Using this and the fact that $x_\ell \in f_{\mathbf{i}^\ell|_K}(I)$ we obtain that $f_{\mathbf{i}^\ell|_K}(I) \subset J^\ell$. It follows that for any $u \in [N]$

$$f_{\mathbf{i}^\ell|_K}(J^u) \subset J^\ell.$$

Hence, there exists an $\mathbf{a}^u \in [L]^K$ such that $f_{\mathbf{i}^\ell|_K}(J^u) = J_{\mathbf{a}^u}^\ell$. It follows that $A_{\mathbf{i}^\ell|_K}(\ell, u) > 0$ and thus for every ℓ and u

$$A^K(\ell, u) = \sum_{i_1, \dots, i_K} A_{i_1, \dots, i_K}(\ell, u) > 0.$$

□

Lemma 5.3. *$\tilde{\eta}$ is σ -invariant and mixing.*

Proof. The proof is a slightly modified version of the proof in [3, Lemma 3.4]. In our case instead of using [3, Lemma 3.1] we use Lemma 5.2. □

Lemma 5.4. $\dim_{\mathbb{H}} \tilde{\eta} < 1$.

Proof. The fact that $\dim_{\mathbb{H}} \nu < 1$ holds was made explicit first in [20, Theorem 44] and the proof of this fact is also available in [2, Section 4.4.4]. On the other hand, $\dim_{\mathbb{H}} \tilde{\eta} = \dim_{\mathbb{H}} \nu$ follows from the arguments in [17, page 357]. □

Recall that we denoted $[M]^{\mathbb{N}}$ by $\Sigma^{(M)}$. Moreover, $\Sigma^{(M)}$ is equipped with the metrics

$$\rho(\mathbf{i}, \mathbf{j}) := L^{-|\mathbf{i} \wedge \mathbf{j}|}.$$

For any $U \subset \Sigma^{(M)}$ we have

$$\overline{\dim}_B(U) = \lim_{n \rightarrow \infty} \frac{\log \#\mathcal{G}_n(U)}{n \log L},$$

for

$$\mathcal{G}_n(U) = \{(j_1, \dots, j_n) \in [M]^n : U \cap [j_1, \dots, j_n] \neq \emptyset\}.$$

Let $\tau : \Sigma^{(L)} \rightarrow \Sigma^{(M)}$,

$$\tau(\mathbf{a}) := (\Pi^{(M)})^{-1} \left(\bigcup_{\ell \in [N]} \Xi_\ell(\mathbf{a}) \right).$$

Now we briefly explain the meaning of this function. Recall that Ξ_ℓ is the composition of a projection from the symbolic space $\Sigma^{(L)}$ to $[0, L]$ and a translation to the ℓ -th interval, J^ℓ . Hence, what we do with τ is that we take an L -adic representation $\mathbf{a} \in \Sigma^L$, then consider the point x with L -adic representation \mathbf{a} and translate it with the left-endpoints of the intervals J^1, \dots, J^N to get the points $x_1, \dots, x_N \in \mathbb{R}$. Then consider the symbolic space corresponding the iterated function system, Σ^M . The resulting set consists of those points of Σ^M which natural projection $(\Pi^{(M)})$ is one of x_1, \dots, x_N .

The relations between the different projections are indicated by the following commutative diagram.

$$\begin{array}{ccc} \Sigma^{(M)} & \xleftarrow{\tau(\cdot)} & \Sigma^{(L)} \\ \Pi^{(M)}(\cdot) \downarrow & \swarrow \Xi_k(\cdot) & \downarrow \Pi^{(L)}(\cdot) \\ J^k & \xleftarrow{t_k(\cdot)} & [0, L] \end{array}$$

In what follows our goal is to upper bound the upper box dimension of the set $\tau(\mathbf{a})$ for a large set of points $\mathbf{a} \in \Sigma^{(L)}$.

Recall that for any $\mathbf{a} \in \Sigma^{(L)}$:

$$\overline{\dim}_B(\tau(\mathbf{a})) = \limsup_{n \rightarrow \infty} \frac{\log \#\mathcal{G}_n(\tau(\mathbf{a}))}{n \log L}.$$

In what follows we give an upper bound on $\#\mathcal{G}_n(\tau(\mathbf{a}))$.

Lemma 5.5. *For any $\mathbf{a} = (a_1, \dots, a_n, \dots)$ we have that $\#\mathcal{G}_n(\tau(\mathbf{a})) \leq \|A_{a_1, \dots, a_n}\|$.*

Proof. Fix $\mathbf{a} \in \Sigma^{(M)}$. From the definition of $\mathcal{G}_n(\tau(\mathbf{a}))$ it follows that for a $(j_1, \dots, j_n) \in [M]^n$ we have $(j_1, \dots, j_n) \in \mathcal{G}_n(\tau(\mathbf{a}))$ if there exists an $\mathbf{i} \in \Sigma^{(M)}$ with $\mathbf{i}|_n = j_1, \dots, j_n$ and an $\ell \in [N]$ such that $\lim_{k \rightarrow \infty} f_{\mathbf{i}|_k}(0) = \Xi_\ell(\mathbf{a}) \in J_{a_1, \dots, a_n}^\ell$. It follows that $\Xi_\ell(\mathbf{a}) \in f_{j_1 \dots j_n}(\Lambda) \subset f_{j_1 \dots j_n}(\cup_{k \in [N]} J^k)$. Therefore there exists a k such that $\Xi_\ell(\mathbf{a}) \in f_{j_1 \dots j_n}(J^k)$. Since for any interval J^k there exists ℓ and $\mathbf{b} = (b_1, \dots, b_n)$ such that $f_{j_1, \dots, j_n}(J^k) = J_{\mathbf{b}}^\ell$, it follows from $J_{\mathbf{a}|_n}^\ell \cap f_{j_1 \dots j_n}(J^k) \neq \emptyset$, that $J_{\mathbf{a}|_n}^\ell = f_{j_1 \dots j_n}(J^k)$. Let

$$U_n^{k, \ell}(\mathbf{a}) := \# \left\{ (j_1, \dots, j_n) \in [M]^n : f_{j_1, \dots, j_n}(J^k) = J_{\mathbf{a}|_n}^\ell \right\}.$$

It is easy to see from the definition of A_{a_1, \dots, a_n} , the argument presented above and the definition of $U_n^{k, \ell}(\mathbf{a})$

$$\begin{aligned} & \# \{(j_1, \dots, j_n) \in [M]^n : [j_1 \dots j_n] \cap \tau(\mathbf{a}) \neq \emptyset\} \\ & \leq \# \{(j_1, \dots, j_n) \in [M]^n : \exists k, \exists \ell f_{j_1, \dots, j_n}(J^k) = J_{a_1, \dots, a_n}^\ell\} \\ & \leq \sum_{k \in [N]} \sum_{\ell \in [N]} U_n^{k, \ell}(\mathbf{a}) = \|A_{a_1, \dots, a_n}\|. \end{aligned}$$

□

It follows from Lemma 5.5, that

$$\overline{\dim}_B(\tau(\mathbf{a})) \leq \limsup_{n \rightarrow \infty} \frac{\log(\|A_{a_1, \dots, a_n}\|)}{n \log(L)}.$$

Recall from 3.13 the pressure function $P(t)$ as

$$P(t) := \lim_{n \rightarrow \infty} \frac{1}{n \log(L)} \log \sum_{\substack{(a_1, \dots, a_n) \\ \in [L]^n}} \|A_{a_1, \dots, a_n}\|^t.$$

Since $\#[L]^n = L^n$, it is easy to see, that $P(0) = 1$. Also, $\sum_{(a_1, \dots, a_n) \in [L]^n} (\|A_{a_1, \dots, a_n}\|) = \|A^n\|$ and from the meaning of the matrices A and A^n , it follows that $P(1) = \frac{n \log M}{n \log L} = s$.

Lemma 5.6. *The function $t \rightarrow P(t)$ exists for $t \in \mathbb{R}$, monotone increasing, convex, continuous, and continuously differentiable for $t > 0$.*

Proof. Since the matrix A is primitive (see Lemma 5.2), the proof is similar to the proof of Lemma 4.3 in [3].

□

The following Lemma is also a slightly modified version of Lemma 4.5. in [3].

Lemma 5.7. *There exists a unique ergodic, shift invariant Gibbs measure μ_1 on $\Sigma^{(L)}$ such that,*

- (1) *there exists a $C > 0$ that for any $(a_1, \dots, a_n) \in [L]^n$ we have $C^{-1} \leq \frac{\mu_1((a_1, \dots, a_n))}{\|A_{a_1, \dots, a_n}\| M^{-n}} \leq C$;*
- (2) $\dim_{\text{H}} \mu_1 = -P'(1) + s$;
- (3) $\lim_{n \rightarrow \infty} \frac{\log \|A_{a_1, \dots, a_n}\|}{n \log(L)} = P'(1)$ for μ_1 almost every $\mathbf{a} \in \Sigma^{(L)}$.

From Lemma 5.3 we know that $\tilde{\eta}$ is also an ergodic measure on $\Sigma^{(L)}$. Hence, it follows that $\tilde{\eta} = \mu_1$.

Thus by Lemma 5.4 $\dim_{\text{H}} \mu_1 = \dim_{\text{H}} \tilde{\eta} < 1$, so by the second part of Lemma 5.7, $P'(1) > s - 1$. Hence, it follows that there exists a $t' \in [0, 1]$, such that

$$(5.2) \quad P(t') < 1 + (s - 1)t'.$$

See Figure 14 for a visual explanation. We can choose a $\delta > 0$ in the way that

$$P(t') - (s - 1 - \delta)t' < 1, \text{ hence } P^*(s - 1 - \delta) < 1,$$

since $s = \frac{\log M}{\log L}$ this completes the proof of Proposition 3.15 with the choice of $\alpha = s - 1 - \delta$. □

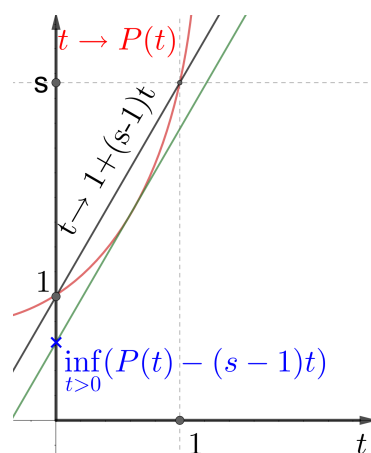


FIGURE 14. Explanation of (5.2).

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