I. HELYETTESÍTÉS

12.(?) gyak

$$\int f(x)\,dx \underset{x=u(t)}{=} \left(\int f(u(t))\cdot u'(t)\,dt\right)_{t=u^{-1}(x)}$$

Pl. $\displaystyle\int e^{-t^2}(-2t)\,dx \underset{\substack{x=-t^2\\ \frac{dx}{dt}=-2t}}{=} \int e^{x}dx = e^{x} = e^{-t^2}$

A gyakorlatban:

① $\displaystyle\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2(t)}\,\cos(t)\,dt = \int \cos^2(t)\,dt = \int \frac{\cos(2t)+1}{2}\,dt =$

$x = \sin(t)$

$\frac{dx}{dt} = \cos(t)$

$\displaystyle = \frac{\sin(2t)}{4} + \frac{1}{2}t + C = \frac{2\sin(t)\cos(t)}{4} + \frac{1}{2}t + C = \frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\arcsin(x) + C$

② $\displaystyle\int (5x+3)^3\,dx = \int u^3\cdot\frac{1}{5}\,du = \frac{u^4}{20} + C = \frac{(5x+3)^4}{20} + C$

$u = 5x+3 \quad x = \frac{u-3}{5}$

$\frac{du}{dt} = \frac{1}{5}$

a) $\displaystyle\int \frac{1}{\sqrt{(1-x^2)}^{\,3}}\,dx$ b) $\displaystyle\int \frac{1}{x\sqrt{x^2+a^2}}\,dx$ c) $\displaystyle\int \sin(x)\cos(x)\,dx$

d) $\displaystyle\int \sqrt{1+x^2}\,dx =$ e) $\displaystyle\int \sqrt{e^x-1}\,dx =$ f) $\displaystyle\int \sqrt{\frac{x}{x-1}}\,dx =$

g) $\displaystyle\int \frac{\sqrt{x}}{x(x+1)}\,dx =$

H: a) $\sin(t) = x$ b) $t = a/x$ c) $u = \sin(x)$ d) $x = \mathrm{sh}(t)$ e) $\sqrt{e^x-1} = t$

f) $\sqrt{x} = \mathrm{ch}(t)$ g) $u = \sqrt{x}$

a) $\int \frac{1}{\sqrt{(1-x^2)^3}} = \int \frac{1}{\sqrt{1-\sin^2(t)}^3} \cos(t)\, dt = \int \cos(t)^{-2} dt = tg(t) + c$

$= tg(\arcsin(x)) + c = \frac{x}{\sqrt{1-x^2}} + c$

$x = \sin(t)$

$\frac{dx}{dt} = \cos(t)$

$\cos^2(\arcsin(x)) + \sin^2(\arcsin(x)) = 1$

$\cos(\arcsin(x)) = \sqrt{1-\sin^2(x)}$

b) $\int \frac{1}{x\sqrt{x^2+a^2}} dx = \int \frac{1}{x^2\sqrt{1+a^2/x^2}} dx = \int \frac{-a}{t^2 \cdot \frac{a^2}{t^2}\sqrt{1+t^2}} dt = -\frac{1}{a}\int \frac{1}{\sqrt{1+t^2}} dt =$

$t = a/x \Rightarrow x = a/t$

$\frac{dx}{dt} = -a/t^2$

$= -\frac{1}{a} \operatorname{arsh}(t) + C =$

$= -\frac{1}{a} \cdot \operatorname{arsh}(a/x) + c$

c) $\int \sin(x)\cos(x)\, dx = \int u\, du = \frac{u^2}{2} = \frac{\sin^2(x)}{2} + c$ ☺

$u = \sin(x)$

$x = \arcsin(u)$

$\frac{dx}{du} = \frac{1}{\sqrt{1-u^2}}$

$\cos(x) = \cos(\arcsin(u)) = \sqrt{1-u^2}$

d) $\int \sqrt{1+x^2}\, dx = \int \sqrt{1+sh^2(t)}\, ch(t)\, dt = \int ch^2(t)\, dt = \int \frac{ch(2t)+1}{2} dt =$

$x = sh(t)$

$\frac{dx}{dt} = \cosh(t)$

$= \frac{sh(2t)}{4} + \frac{1}{2}t + c =$

$= \frac{1}{2}x \cdot \sqrt{1+x^2} + \frac{1}{2}\operatorname{arsh}(x) + C$

e) $\int \sqrt{e^x - 1}\, dx = \int \frac{2t^2}{t^2+1} dt = 2\int \frac{t^2+1-1}{t^2+1} dt = 2 \cdot \int 1 - \frac{1}{t^2+1} dt$

$t = \sqrt{e^x-1} \Rightarrow t^2 = e^x - 1 \Rightarrow x = \ln(t^2+1)$

$\frac{dx}{dt} = \frac{1}{t^2+1} \cdot 2t$

$= 2 \cdot t - 2 \cdot arctg(t) + C =$

$= 2 \cdot \left[\sqrt{e^x-1} - arctg(\sqrt{e^x-1})\right] + C$

f) $\int \sqrt{\frac{x}{x-1}}\,dx = \int \frac{ch(t)}{\sqrt{ch^2(t)-1}}\, 2ch(t)sh(t)\,dt =$

$\sqrt{ch^2(t)-1} = sh(t)$

$\sqrt{x} = ch(t)$ ($\Rightarrow t = arch(\sqrt{x})$)

$x = ch^2(t)$

$\frac{dx}{dt} = 2ch(t)\,sh(t)$

$= \int 2ch^2(t)\,dt = \int ch(2t)+1\,dt =$

$= \frac{sh(2t)}{2} + t + C =$

$= sh(t)ch(t) + t + C =$

$= \sqrt{x}\,\sqrt{x-1} + arch(\sqrt{x}) + C$

g) $\int \frac{\sqrt{x}}{x(x+1)}\,dx = \int \frac{u}{u^2(u^2+1)} \cdot 2u\,du = \int \frac{2u^2}{u^2(u^2+1)}\,du =$

$u = \sqrt{x}$

$x = u^2$

$\frac{dx}{du} = 2u$

$= \int \frac{2}{u^2+1}\,du = 2 \cdot arctg(u) + C$

$= 2\,arctg(\sqrt{x}) + C$

RACIONÁLIS TÖRT FGVEK

MOTIVÁCIÓ

$$\int \frac{2x^5+3x^3-2x^2-x+1}{x^2+x+2}\,dx = ☺$$

1. lépés Polinom osztás:

$$3x^5+0x^4+x^3-4x^2-x+1 : x^2+x+3 = 3x^3-3x^2-5x+10$$

$$\begin{array}{l}
-(3x^5+3x^4+9x^3) \\
\hline
0 - 3x^4-8x^3-4x^2 \\
\quad -(3x^4-3x^3-9x^2) \\
\hline
\quad 0 \quad -5x^3+5x^2-x \\
\quad -(-5x^3-5x^2-15x) \\
\hline
\qquad 0 \quad 10x^2+14x+1 \\
\qquad -(10x^2+10x+30) \\
\hline
\qquad 0 \quad \underline{4x-29}
\end{array}$$

JELENTÉS:

$$3x^5+x^3-4x^2-x+1 = (x^2+x+3)\cdot(3x^3-3x^2-5x+10)+4x-29,$$

tehát:

$$☺ = \int \frac{(x^2+x+3)\cdot(3x^3-3x^2-5x+10)+4x-29}{(x^2+x+3)}\,dx =$$

$$= \int 3x^3-3x^2-5x+10+\frac{4x-29}{(x^2+x+3)}\,dx =$$

$$= \int 3x^3-3x^2-5x+10\,dx + \int \frac{4x-29}{x^2+x+3}\,dx =$$

$$= \int 3x^3 - 3x^2 - 5x + 10 \, dx =$$

$$= \frac{3x^4}{4} - \frac{3x^2}{3} - \frac{5x^2}{2} + 10x + c$$

$$= \int \frac{4x-29}{x^2+x+3} \, dx =$$

x^2+x+3 diszkriminánsa $1-12=-11<0$

miért fontos a diszkrimináns?

ax^2+bx+c -re $\quad D = b^2-4ac<0 \Rightarrow$

átírható $\quad A\left[\left(\frac{x-B}{C}\right)^2+1\right]$ alakba

← teljes négyzet

Biz: $\quad ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) =$

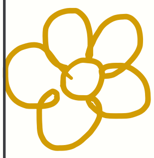
$$= a\left(x^2+2\cdot\frac{b}{a}\cdot\frac{1}{2}x+\left(\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\left(\frac{b}{2a}\right)^2\right)\right)=$$

$$= a\left(\left[x+\frac{b}{2a}\right]^2+\frac{4ac-b^2}{2a^2}\right) =$$

$$\frac{4ac-b^2}{2a^2} = \frac{-D}{2a^2} > 0 \Rightarrow$$ gyököt lehet vonni!

$$= a\cdot\frac{4ac-b^2}{2a^2}\left[\left\{\left(x+\frac{b}{2a}\right)\left(\frac{2a^2}{4ac-b^2}\right)^{\frac{1}{2}}\right\}^2+1\right]$$

Ha a diszkrimináns $\geq 0 \Rightarrow$ parciális törtekre bontás! (lásd később)

 $= \displaystyle\int \frac{4x-29}{x^2+x+3}\,dx = 2\cdot\int \frac{2x+1-\frac{31}{2}}{x^2+x+3}\,dx =$

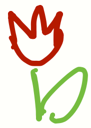
most 2 részre bontjuk a nevezőt:

$Q(x) = x^2+x+3 \Rightarrow Q'(x) = 2x+1$

$\alpha\cdot\displaystyle\int \frac{Q'(x)}{Q(x)}\,dx + \int \frac{k}{Q(x)}\,dx$ vmi k konstansra.

$\alpha\cdot\ln|Q(x)| + C$

$\hookrightarrow$ diszkr. $(Q(x)) < 0$ $\Downarrow$

 miatt

$B = \displaystyle\int \frac{8x+3}{x^2+x+3}\,dx = 4\int \frac{(2x+1)-1}{x^2+x+3}\,dx =$

$= 4\cdot\ln|x^2+x+3| + \quad \ldots$

$4\cdot\displaystyle\int \frac{1}{x^2+2\cdot\frac{1}{2}x+\frac{1}{4}+\frac{11}{12}}\,dx = 4\cdot\int \frac{1}{(x+\frac{1}{2})^2+\frac{11}{12}}\,dx =$

$= 4\cdot\frac{12}{11}\cdot\displaystyle\int \frac{1}{\left[\left(x+\frac{1}{2}\right)\left(\sqrt{12}/\sqrt{11}\right)\right]^2+1}\,dx = 4\cdot\frac{12}{11}\cdot\operatorname{arctg}\left(\frac{x+\frac{1}{2}}{\sqrt{11}/\sqrt{12}}\right)\cdot\frac{\sqrt{12}}{\sqrt{11}}$

Polinomosztás feladatok

a) $x^5 - x^4 + x^3 - x^2 + x - 1 : x^4 + x^2 + 1 = x - 1$

$$-(x^5 + 0x^4 + x^3 + 0x^2 + x) \downarrow$$

$$0 - x^4 + 0 - x^2 + 0 - 1$$

$$-(-x^4 + 0x^3 - x^2 \quad - 1)$$

$$0$$

b) $x^k + 0x^{k-1} + \dots - 1 \quad : x - 1 = x^{k-1} + x^{k-2}$

$$-(x^k - x^{k-1})$$

$$x^{k-1} + 0x^{k-2} + \dots - 1$$

$$-(x^{k-1} - x^{k-2})$$

$x^{k-2} + \dots + -1$ $\Rightarrow$ ez így megy tovább $\Rightarrow$ fejtés: $x^k - 1 : x - 1 = x^{k-1} + x^{k-2} + \dots + x$

Ell.

$$(x-1) \cdot (x^{k-1} + \dots + 1) = x^k + x^{k-1} + x^{k-2} + \dots + x$$
$$- x^{k-1} - x^{k-2} - \dots - x - 1 =$$
$$= x^k - 1$$

c) $x^6 - x^5 + x^2 + 2x + 3 : x^3 - x^2 = x^3$

$$-(x^6 - x^5) \downarrow$$

$$0 - 0 + x^2 + 2x + 3$$

$$\Rightarrow (x^3 - x^2) \cdot x^3 + x^2 + 2x + 3 = x^6 - x^5 + x^2 + 2x + 3$$

Parciális törtekre bontás

Motiváció:

$$\int \frac{(x-3)}{(x-1)(x-2)}\,dx = ?$$

$$\int \frac{(x-3)}{(x-1)(x-2)}\,dx = \int \frac{A}{(x-1)} + \frac{B}{(x-2)}\,dx = A\ln(|x-1|) + B\ln(|x-2|) + \text{const.}$$

Mi lesz A és B?

$$\frac{(x-3)}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} = \frac{A(x-2)+B(x-1)}{(x-2)(x-1)} =$$

$$= \frac{Ax-2A+Bx-B}{(x-2)(x-1)}$$

$$x\cdot 1 + 1(-3) = x(A+B) + 1(-2A-B)$$

tek. $\frac{P(x)}{Q(x)}$, $\deg(P(x)) < \deg(Q(x))$,

ha $Q(x) = a_n\cdot x^n + \dots + a_1 x + a_0$, és

$$Q(x) = a_n(x-x_0)^{\alpha_0}\cdot\ldots\cdot(x-x_m)^{\alpha_m}(x^2+b_1x+c_1)^{\beta_1}\cdot\ldots\cdot(x^2+b_\ell x+c_\ell)^{\beta_\ell}$$

$$\frac{P(x)}{Q(x)} = \frac{A_1^{(1)}}{(x-x_0)} + \dots + \frac{A_{\alpha_0}^{(1)}}{(x-x_0)^{\alpha_0}} + \frac{A_1^{(2)}}{(x-x_1)} + \dots + \frac{A_{\alpha_1}^{(2)}}{(x-x_1)^{\alpha_1}} + \dots + \frac{A_1^{(m)}}{(x-x_m)} + \dots + \frac{A_{\alpha_1}^{(m)}}{(x-x_m)^{\alpha_m}} +$$

$$+ \frac{B_1^{(1)}x + C_1^{(1)}}{x^2+b_1x+c_1} + \dots + \frac{B_{\beta_1}^{(1)}x + C_{\beta_1}^{(1)}}{x^2+b_1x+c_1} + \dots + \frac{B_\ell^{(\ell)}x + C_\ell^{(\ell)}}{x^2+b_\ell x+c_\ell} + \dots + \frac{B_{\beta_\ell}^{(\ell)}x + C_{\beta_\ell}^{(\ell)}}{x^2+b_\ell x+c_\ell} +$$

Példák

(1.) $\displaystyle\int \frac{x-7}{(x-1)^2\cdot(x-3)}\,dx =$ (2.) $\displaystyle\int \frac{x^2+5}{(x^2+x+2)(x-3)}\,dx =$

(1.) $$\frac{x-7}{(x-1)^2\cdot(x-3)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x-3)} =$$

$$= \frac{A(x-3) + B(x^2-4x+3) + C(x^2-2x+1)}{(x-1)^2(x-3)} =$$

$$= \frac{x^2\cdot(B+C) + x(A-4B-2C) + 1(-3A+3B+C)}{(x-1)^2(x-3)}$$

$\Downarrow$

$B+C=0$ $\qquad B=-C$

$A-4B-2C=1 \Rightarrow A-4B+2B = A-2B = 1 \Rightarrow A = 2B+1$

$-3A+3B+C=-7 \Rightarrow -3(2B+1)+3B-B=-7$

$\Downarrow$

$-6B-3+2B=-7$

$-4B=-4 \Rightarrow B=1 \Rightarrow A=3$

$\Rightarrow C=-1$

$$\frac{x-7}{(x-1)^2\cdot(x-3)} = \frac{3}{(x-1)^2} + \frac{1}{(x-1)} + \frac{-1}{(x-3)}$$

2. $$\frac{x^2+5}{(x^2+x+2)(x-3)} = \frac{Ax+B}{x^2+x+2} + \frac{C}{x-3} =$$

$$= \frac{Cx^2+Cx+2C+Ax^2+Bx-3Ax-3B}{(x^2+x+2)(x-3)} =$$

$$= \frac{x^2(C+A)+x(C+B-3A)+1(2C-3B)}{(x^2+x+2)(x-3)}$$

$$\begin{aligned} C+A&=1 \\ C+B-3A&=0 \\ 2C-3B&=5 \end{aligned} \Rightarrow \begin{aligned} A&=0 \\ B&=-1 \\ C&=1 \end{aligned}$$

$$\Rightarrow \frac{x^2+5}{(x^2+x+2)(x-3)} = \frac{-1}{x^2+x+2} + \frac{1}{x-3}$$

Feladatok: Integráljuk az alábbiakat

a) $\frac{x}{x^3-1}$ b) $\frac{1}{x^2-2x-3}$ $\boxed{c)}$ $\frac{1}{(x+1)^2(x^2+1)}$

$\boxed{d)}$ $\frac{8}{7x+3x^5}$ (helyettesítünk $t=x^4$)

$\boxed{e)}$ $\frac{x^3-2x^2+4}{x^3(x-2)^2}$

ezekhez a kidolgozott mo még hiányzik

a) $\int \frac{x}{x^3-1}\,dx = ☺$

$$\frac{x}{x^3-1} = \frac{x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} =$$

$$= \frac{Ax^2 + Ax + \boxed{A} + Bx^2 - \boxed{B} + Cx - \boxed{C}}{x^3-1}$$

$$0\cdot x^2 + 1\cdot x + 0\cdot 1 = x^2(A+B) + x(A+C) + 1\cdot(A-B-C)$$

$0 = A + B \Rightarrow B = -A$

$1 = A + C \Rightarrow C = 1 - A$

$0 = A - B - C = A + A + A - 1 \Rightarrow 3A = 1$

$A = 1/3$

$C = 2/3$ $\qquad B = -1/3$

$$☺ = \boxed{\frac{1/3}{x-1} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2+x+1}}$$

$$\int \frac{1/3}{x-1} + \frac{-1/3\,x + 2/3}{x^2+x+1}\,dx =$$

$$= \underbrace{\frac13 \int \frac{1}{x-1}\,dx}_{} + \underbrace{\int \frac{-1/3\,x + 2/3}{x^2+x+1}\,dx}_{} = \star$$

$$= \frac13 \ln|x-1| + c$$

$$= -\frac13 \int \frac{x-2}{x^2+x+1}\,dx = -\frac13\cdot\frac12 \int \frac{2x+1 - 5/2}{x^2+x+1}\,dx =$$

$$= \underline{-\frac13\,\frac12 \cdot \ln|x^2+x+1| + c} + \left(+\frac13\cdot\frac12\cdot\frac52\,\boxed{\int \frac{1}{x^2+x+1}\,dx}\right)$$

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{(x+\frac12)^2+\frac34}\,dx = \frac{1}{3/4}\cdot\int \frac{1}{[(x+\frac12)/(\sqrt3/2)]^2+1}\,dx =$$

$$x^2 + 2\cdot\frac12\cdot x + \frac14 + \frac34 = x^2+x+1$$

$$\| $$

$$(x+\frac12)^2 + \frac34$$

$$\int \frac{1}{1+x^2}\,dx = \arctg(x) + c$$

$$F' = f \qquad \int f(ax+b)\,dx = \frac{F(ax+b)}{a} + c$$

$$= \frac43 \cdot \int \frac{1}{\underbrace{[2/\sqrt3\,(x+\frac12)]^2}_{} + 1}\,dx = \frac43 \arctg\left([2/\sqrt3\,(x+\tfrac12)]^2\right)\cdot\frac{\sqrt3}{2} + c =$$

$$= \frac{2}{\sqrt3} \arctg\left([2/\sqrt3\,(x+\tfrac12)]^2\right) + c$$

$$\bigstar = \frac{1}{3} \ln|x-1| + c$$

$$-\frac{1}{3} \cdot \frac{1}{2} \cdot \ln|x^2+x+1| + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{5}{2} \left[\frac{2}{\sqrt{3}} \operatorname{arctg}\left(\left[\frac{2}{\sqrt{3}} \left(x+\frac{1}{2}\right) \right]^2 \right) + c \right] + C$$

b) $$\frac{1}{x^2-2x-3} = \frac{1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3} =$$

$$= \frac{Ax - 3A + Bx + B}{(x+1)(x-3)} \Rightarrow$$

$$\left.\begin{aligned} A+B &= 0 \\ -3A+B &= 1 \end{aligned}\right\} \quad B = \frac{1}{4} \Rightarrow A = -\frac{1}{4}$$