MANDELBROT PERCOLATION AND SOME OF ITS GENERALIZATIONS: A SURVEY.

VILMA ORGOVÁNYI

HUN-REN–BME Stochastics Research Group, Műegyetem rkp. 3., H-1111 Budapest, Hungary
Department of Stochastics, Institute of Mathematics, Budapest University of Technology and Economics, Műegyetem rkp. 3., H-1111 Budapest, Hungary

KÁROLY SIMON

HUN-REN–BME Stochastics Research Group, Műegyetem rkp. 3., H-1111 Budapest, Hungary
Alfréd Rényi Institute of Mathematics – Eötvös Loránd Research Network, Reáltanoda u. 13- 15., 1053 Budapest, Hungary

Abstract. In this survey we collect some results regarding the Mandelbrot percolation fractal and some of its natural generalizations. We discuss projections and connectivity in detail. We also touch upon porosity, visibility, differences of independent Mandelbrot percolations (i.e. projections of products of independent percolations) and Assouad type dimensions. In the Appendix, some more general constructions are explored.

Contents

E-mail addresses: orgovanyi.vilma@gmail.com, karoly.simon51@gmail.com.
VO was supported by the grant KKP144059 "Fractal geometry and applications".
KS was supported by the grant NKFI K142169.

1. Introduction

The aim of this survey is to collect some results regarding the Mandelbrot percolation fractal and some of its most natural generalizations.

Definition 1.1. The Mandelbrot percolation (fractal percolation) set is a two-parameter family of random sets in $\mathbb{R}^d$. Namely, fix the parameters $K \geq 2$ and $p \in (0,1]$. In the first step of the construction we partition the unit cube $[0,1]^d$ into $K^d = M$ axes-parallel cubes of side length $1/K$. Each of these cubes are retained with probability p and discarded with probability $1-p$ independently. This step is repeated independently in each of the retained cubes *ad infinitum* or until no retained cubes are left. The resulting random (compact) set is the Mandelbrot percolation set, and we denote it by $\Lambda(K,p)^d = \Lambda(K,p)$.

In other words we run a percolation process on the cylinders of a special grid-aligned IFS. When the percolation process runs on the full grid and all the probabilities agree as above, we call this *homogeneous* Mandelbrot percolation. The model where for each first level cylinder the chosen probabilities may differ (but remain the same between different levels) is called *inhomogeneous* Mandelbrot percolation; see Section 2 for a more detailed description. An important special case occurs when we run the process on a proper subset of the grid (such as in the construction of the random Sierpiński carpet, see Figure 1b). In this case the retention probability of each square is p, except for the middle square which we retain with probability 0. In fact, we will run the percolation process on the cylinder sets of arbitrary iterated function systems; see (1.3). We refer to such random systems as *coin tossing systems*, since we decide which cylinders to retain and which to discard based on the outcome of repeated independent coin tosses.

The structure of the survey is as follows:

- In the remaining part of the introduction we define the model we will work with, state some preliminary result, and then overview results concerning the Mandelbrot percolation fractal.
- The body of the survey starts with some early results about *projections of grid-aligned IFSs to the coordinate axes* (in this case the cylinders of the projected IFSs have either insignificant or exact overlaps). We describe results about the Hausdorff and box-dimension of the projected sets, as well as the existence of interior points in such sets and the positivity of their Lebesgue measure.
- We then explain some recent result about *rational projections of grid-aligned IFSs* (and a slight generalization thereof). We consider positivity of Lebesgue measure and existence of interior points in these cases, and briefly mention some known results concerning the dimension of such sets.

- Next, we move on to results about *projections of (homogeneous) Mandelbrot percolation to any direction.*
- Finally, we summarize results about percolation—the existence and properties of connected components in the homogeneous Mandelbrot percolation set which connects the left and the right sides of the unit quare.
- In the appendix we mention two of the more general models of random sets and measures: the Mandelbrot cascades and substitution random fractals.

1.1. **The model.** Fix a self-similar IFS $\mathcal{F}$ on $\mathbb{R}^d$,

$$\mathcal{F} := \{f_i(x) := r_i Q_i x + t_i\}_{i=0}^{M-1}, \; f_i : \mathbb{R}^d \to \mathbb{R}^d, \; r_i \in (0,1), \; Q_i \in O(d), \; t_i \in \mathbb{R}^d. \tag{1.1}$$

We use the shorthand notation

$$[M] := \{0, \dots, M-1\},$$

and for $\mathtt{i} = (i_1, \dots, i_n) \in [M]^n$

$$f_{\mathtt{i}} = f_{i_1,\dots,i_n} := f_{i_1} \circ \cdots \circ f_{i_n}, \; r_{\mathtt{i}} = r_{i_1,\dots,i_n} := r_{i_1} \dots r_{i_n}, \; .$$

Since the maps are contractions, we may fix

$$B \subset \mathbb{R}^d \text{ compact such that } f_i(B) \subset B \quad \text{for all } i \in [M]. \tag{1.2}$$

The level-n cylinders are

$$\{f_{\mathtt{i}}(B)\}_{\mathtt{i} \in [M]^n}, \tag{1.3}$$

and the union of all n-cylinders $\bigcup_{\mathtt{i} \in [M]^n} f_{\mathtt{i}}(B)$ form a nested sequence of compact sets. Their intersection is the attractor

$$\Lambda_{\mathcal{F}} = \Lambda := \bigcap_{n=1}^{\infty} \bigcup_{\mathtt{i} \in [M]^n} f_{\mathtt{i}}(B).$$

The definition of Λ does not depend on the choice of B as long as B satisfies (1.2).

Definition 1.2 (Coin tossing self-similar sets)**.** Let $\mathcal{F} := \{f_i\}_{i=0}^{M-1}$ be a (deterministic) self-similar IFS on $\mathbb{R}^d$ as defined in (1.1) and let $p \in (0,1)$. The corresponding coin tossing self-similar set $\Lambda_{\mathcal{F}}(p)$ is defined as follows: In the first step for every $k \in [M]$ we toss (independently) a biased coin which lands on head with probability p. The random subset $\mathcal{E}_1 \subset [M]$ consists of those $k \in [M]$ for which the coin tossing resulted in head. Assume that we have already constructed $\mathcal{E}_n \subset [M]^n$. Then for every node $\mathtt{i} \in \mathcal{E}_n$ we define (independently of everything) the random set $\mathcal{E}_1^{\mathtt{i}} \subset [M]$ which has the same distribution as $\mathcal{E}_1$. The set of the offspring of $\mathtt{i}$ is defined by $O(\mathtt{i}) = \{\mathtt{i}k \in [M]^{n+1} : k \in X_1^{\mathtt{i}}\}$, where $\mathtt{i}k = (i_1, \dots, i_n, k)$ if $\mathtt{i} = (i_1, \dots, i_n)$. Finally, we set $\mathcal{E}_{n+1} = \bigcup_{\mathtt{i} \in \mathcal{E}_n} O(\mathtt{i}) \subset [M]^{n+1}$. Then the coin tossing self-similar set is defined by

$$\Lambda_{\mathcal{F}}(p) = \Lambda(p) := \bigcap_{n=1}^{\infty} \bigcup_{\mathtt{i} \in \mathcal{E}_n} f_{\mathtt{i}}(B),$$

where B is chosen as in (1.2).

For later usage for $\mathtt{i} \in [M]^n$ we further define $\mathcal{E}_k^{[\mathtt{i}]} = \mathcal{E}_{n+k}^{[\mathtt{i}]} \cap [\mathtt{i}]$, the level k retained children of $\mathtt{i}$ (given that $\mathtt{i}$ itself is retained).

This above construction is a special case of the substitution model (see Appendix B) which is a special case of Falconer and Jin's construction in [23] which is described in more detail in Appendix A.

1.2. **Extinction.** It is immediate from the definition of coin tossing systems—running on an IFS with M maps with probability parameter p—that $\#\mathcal{E}_n$ (the number of retained level n cylinders) is a Galton-Watson process with offspring distribution $\mathrm{Bin}(M,p)$. From this, it follows that if $p \leq 1/M$ then the process dies out almost surely, and if $p > 1/M$ then with positive probability the random attractor survives. (The probability of survival is given by $1-q$, where q is the smallest non-negative fixed point of the probability generating function of $\mathrm{Bin}(M,p)$.) In what follows we will always assume that

$$p > \frac{1}{M}, \tag{1.4}$$

so with positive probability the random attractor is not empty.

1.3. **Overview.** The Mandelbrot percolation fractal had been researched by many authors because of its interesting and versatile nature.

For surveys on the subject see [39] by Rams and Simon, focusing on projections of inhomogeneous Mandelbrot percolations and the connected question of sums and differences of independent percolations. For a different perspective on inhomogeneous Mandelbrot percolations and differences of independent percolations see [12] by Dekking. Finally, in [43] by Simon and Vágó we can see yet another perspective, mainly focusing on exceptional projections of 2-dimensional random inhomogeneous fractal percolations.

We mention here some aspects which we don't describe in detail:

(1) Porosity (which very vaguely speaking describes the sizes of holes we can find in the set) was studied by Berlinkov and Järvenpää in 2019 in [5]. Let μ denote the natural measure (see (4.3)) on the 2-dimensional Mandelbrot percolation. Then for μ almost every point x the following holds:
   - the local upper porosity of μ is 1, and the local porosity of the set at x is 1/2 (the maximal possible value),
   - whereas the lower porosity of μ, equals to the local porosity of the set at x and is equal to 0 (the minimal possible value).

   Before this, Chen, Ojala, Rossi, and Suomala in 2018 ([11]) considered the dimension of subsets of exceptional points (note that this also implies the results of Berlinkov and Järvenpää regarding porosities of sets).
(2) Visibility of Mandelbrot percolation was investigated by Arhosalo, Järvenpää, Järvenpää, Rams and Shmerkin in 2012 (see [1]). The visible parts of the set from a line ℓ are those points of the set which are visible from the line when we look at the set orthogonally from ℓ. In [1] it is shown that if the almost sure box-dimension of the set is greater than one (in which case the projection of the set almost surely contains an interval simultaneously in all directions, so the dimension of visible part of the set is at least 1), then the visible part of the set has equal box and Hausdorff dimension, and is equal to 1 for all lines ℓ not meeting the set.
(3) Sums and differences of independent percolations:
   - Dekking and Simon in 2007 ([16]) gave conditions under which the difference of two independent one-dimensional inhomogeneous Mandelbrot percolation set has empty/non-empty interior. In particular, they proved that for 1-dimensional homogeneous Mandelbrot percolations F_1 and F_2, if $\dim_H F_1 + \dim_H F_2 > 1$, then their difference contains an interval almost surely.

- Positivity of Lebesgue measure of two one-dimensional inhomogeneous Mandelbrot percolation was considered in [31] by Móra, Simon and Solomyak in 2009. They concluded that it is possible that the Lebesgue measure of the difference set is positive, but it has empty interior.
- In 2011 Dekking and Kuijvenhoven continuing [16] considered sharp conditions under which the difference contains an interval almost surely in terms of the lower spectral radii (see [14]).
- Subsequently, Simon and Rams in 2014 ([37]) considered $d \geq 2$ independent (homogeneous) Mandelbrot percolations $E_1, \dots E_d$ and their algebraic sum with coefficients $a_1, \dots, a_d$: $a_1E_1 + \cdots + a_dE_d$ and they proved that if for all i $a_i \neq 0$ and $\sum \dim E_i > 1$ then the weighted algebraic sum above has non-empty interior almost surely conditioned on non-extinction.

(4) Assouad dimension, spectrum (for the definition of these can be found in Jonathan Frasers book on the topic: [24, Section 2.1, 3.3 respectively])
- The Assouad dimension of the d-dimensional Mandelbrot percolation is, almost surely conditioned on non-extinction, as large as possible, that is, exactly d; see [25] by Fraser, Miao and Troscheit or the above mentioned [24, Section 9.4]. In other words the random tree corresponding to the set contains a full (much larger than expected) n-level subtree for arbitrary $n \in \mathbb{N}$.
- On the other hand, the Assouad spectrum is equal to the box-dimension almost surely conditioned on non-extinction (see [26] by Fraser and Yu, [45] by Troscheit, or the book [24, Section 9.4]), meaning that larger than expected subtrees are rare.
- Precise information concerning the scale at which larger-than-expected subtrees appear was given by Banaji, Rutar and Troscheit in [3].

2. Projection to coordinate axes

In this section we study inhomogeneous fractal percolation sets which in $\mathbb{R}^2$ are constructed as follows: We divide the square into K^2 congruent axes parallel squares of size $\frac{1}{K} \times \frac{1}{K}$ but the retention probabilities $\mathbf{p} := \{p_{i,j}\}_{i,j=0}^{K}$ are not necessarily the same of these K^2 squares. More precisely, $p_{i,j}$ is the probability of the retention of the square $Q_{i,j} := \left(\frac{i}{K}, \frac{j}{K}\right) + \left[0, \frac{1}{K}\right] \times \left[0, \frac{1}{K}\right]$, $i, j \in [K]$. We will write $m_{i,j}$ for the expectation of the number of squares retained in column $i \in [K]$. That is

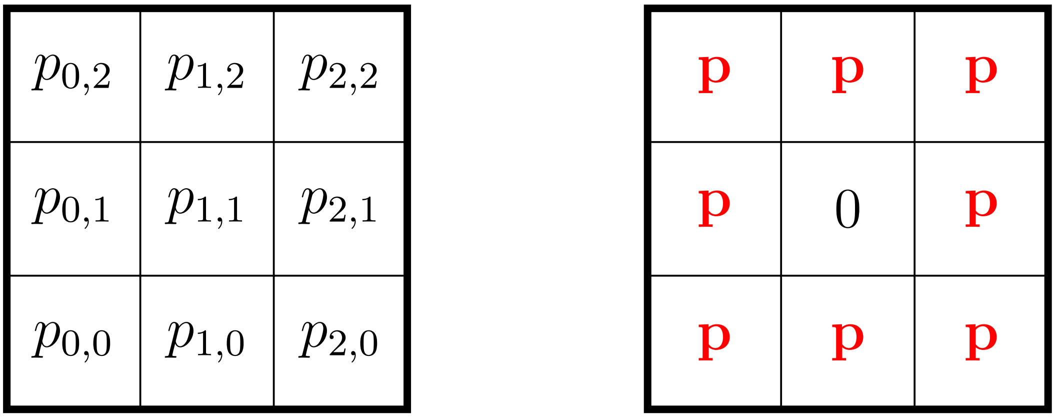
$$m_i := \sum_{j=0}^{K-1} p_{i,j}. \tag{2.1}$$

Let $\Lambda(K, \mathbf{p})$ be the resulting set after infinitely many iterations. Figure 1a shows the extinction probabilities in the case when $K = 3$.

It was proved by Peyrière, Hawkes, Falconer, Mauldin and Williams that

$$\text{if } \Lambda(K, \mathbf{p}) \neq \emptyset \text{ then } \dim \Lambda(K, \mathbf{p}) = \frac{\log \left(\sum_{i,j=0}^{K-1} p_{i,j} \right)}{\log K} = \frac{\log \left(\sum_{i=0}^{K-1} m_i \right)}{\log K} \quad a.s., \tag{2.2}$$

where dim can be both $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{B}}$. The random Sierpiński carpet $\Lambda_{\mathcal{S}}(p)$ is the special case when $K = 3$, and $p_{1,1} = 0$ but for all other $0 \leq i, j \leq 2$, $p_{i,j} = p$ for a $p \in (0, 1)$ (see Figure 1b). Clearly, $\dim_{\mathrm{H}} \Lambda_{\mathcal{S}}(p) = \log(8p)/\log 3$. The box dimension



(A) The inhomogeneous case, $K = 3$ (B) Random Sierpiński carpet

FIGURE 1. In the case of Random Sierpiński carpet, $p_{i,j} = p$ if $(i,j) \neq (1,1)$ and $p_{1,1} = 0$.

part of the following theorem was proved by Dekking and Grimmett [13]. Then Falconer [20] gave a shorter proof which also yielded the Hausdorff dimension part of the following theorem.

Theorem 2.1 (Dekking–Grimmett and Falconer). *Fix $K \geq 2$, and a $\mathbf{p} := \{p_{i,j}\}_{i,j=0}^{M-1}$ with $p_{i,j} \in [0,1]$ such that $\sum_{i,j=0}^{K-1} p_{i,j} > 1$. Let $\Lambda(K,\mathbf{p})$ denote the corresponding inhomogeneous fractal percolation set. Conditioned on non-extinction we have almost surely:*

$$\dim_{\mathrm{H}} \pi_1 \Lambda(K,\mathbf{p}) = \dim_{\mathrm{B}} \pi_1 \Lambda(K,\mathbf{p}) = \frac{\inf\limits_{0 \leq s \leq 1} \left[\log \sum\limits_{i=0}^{K-1} m_i^s \right]}{\log K}, \tag{2.3}$$

where $\pi_1(x,y) := x$ is the orthogonal projection to the x-axis and m_i was defined in (2.1).

The following theorem of Dekking and Grimmett [13, Theorem 1.3] gives conditions under which we have that the dimension of the projection of the inhomogeneous Mandelbrot percolation set onto the x-axis is equal to the dimension of the inhomogeneous Mandelbrot percolation set.

Theorem 2.2. *Using the notation of Theorem 2.1,*

a) *If $\sum_{k=0}^{K-1} m_k \log m_k > 0$ then $\dim \pi_1(\Lambda(K,\mathbf{p})) < \dim(\Lambda(K,\mathbf{p}))$.*

b) *If $\sum_{k=0}^{K-1} m_k \log m_k \leq 0$ then*

$$\dim \pi_1(\Lambda(K,\mathbf{p})) = \dim(\Lambda(K,\mathbf{p})) = \frac{\log\left(\sum_{i,j=0}^{K-1} p_{i,j}\right)}{\log K} \quad a.s.,$$

where dim *is either Hausdorff or box dimension.*

If we apply this for the random Sierpiński carpet $\Lambda_{\mathcal{S}}(p)$ then,

i) if $\frac{1}{8} = 0.125 < p \leq \frac{1}{\sqrt[4]{54}} = 0.368893\ldots$ then $\Lambda_{\mathcal{S}}(p) \neq \emptyset$ with positive probability and $\dim \pi_1 \Lambda_{\mathcal{S}}(p) = \dim \Lambda_{\mathcal{S}}(p)$,

ii) if $\frac{1}{\sqrt[4]{54}} < p \leq \frac{3}{8} = 0.375$ then $\dim \pi_1 \Lambda_{\mathcal{S}}(p) < \dim \Lambda_{\mathcal{S}}(p) \leq 1$.

The following theorem is due to Dekking and Grimmett [13, Theorem 8]. It describes the positivity of the Lebesgue measure of the axis-projection of the inhomogeneous fractal percolation sets.

Theorem 2.3 (Dekking-Grimmett)**.** *Fix a* $K \geq 2$*, and a* $\mathbf{p} := \{p_{i,j}\}_{i,j=0}^{K-1}$ *with* $p_{i,j} \in [0,1]$ *such that for every* j *there exists an* i *with* $p_{i,j} > 0$*. Let* $\Lambda(K, \mathbf{p})$ *be the corresponding inhomogeneous fractal* p *percolation set. For every* $i \in [K]$ *let* $m_i := \sum_{j=0}^{K-1} p_{i,j}$*, and* $m := \prod_{i=0}^{K-1} m_i$*. Then*

a) $\mathcal{L}(\pi_1 \Lambda(K, \mathbf{p})) = 0$ *a.s. if* $m \leq 1$*.*
b) $\mathcal{L}(\pi_1 \Lambda(K, \mathbf{p})) > 0$ *a.s. conditioned on non-extinction if* $m > 1$*.*

Applying this to the random Sierpiński carpet,

1) If $0.375 = \frac{3}{8} < p \leq \frac{1}{\sqrt[4]{18}} = 0.381571\ldots$ then $\dim_{\mathrm{H}} \Lambda_{\mathcal{S}}(p) > 1$ but $\mathcal{L}(\pi_1 \Lambda_{\mathcal{S}}(p)) = 0$ a.s..
2) If $p > \frac{1}{\sqrt[4]{18}}$ then $\mathcal{L}(\pi_1 \Lambda_{\mathcal{S}}(p)) > 0$ a.s..

Concerning the existence of interior points of axis-projections of inhomogeneous Fractal percolation sets Falconer and Grimmett [22] proved the following theorem.

Theorem 2.4 (Falconer, Grimmett)**.** *Assume that in every vertical column there are at least two squares that we can retain with positive probability and the sum of the probabilities of the squares in every vertical column is greater than 1. Then* $\pi_1(\Lambda(K, \mathbf{p}))$ *has non-empty interior a.s. conditioned on non-extinction.*

For the random Sierpiński carpet $\Lambda_{\mathcal{S}}(p)$ this implies that conditioned on non-extinction, the axis-projection $\pi_1(\Lambda_{\mathcal{S}}(p))$ contains interior points if $p > \frac{1}{2}$.

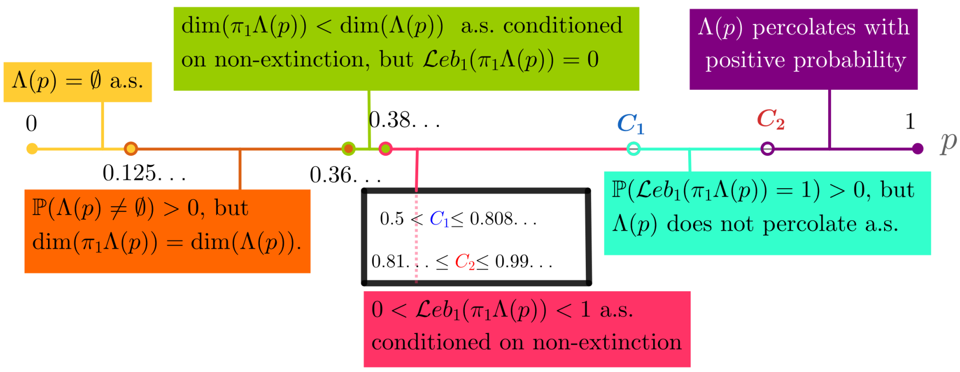


FIGURE 2. The coordinate axis projection of the Random Sierpiński carpet

.

Dekking and Meester [15] extended the results above stated for the random Sierpiński carpet by proving that there is an interval of probabilities where the random Sierpiński carpet percolates. Figure 2 summarizes all these results about the random Sierpiński carpet.

3. Rational projections and special families of finite type

In this section we consider random sets obtained by running the percolation process on systems of the form:

$$\mathcal{F} := \left\{ f_i(x) := \frac{1}{L}x + t_i \right\}_{i=0}^{M-1}, \; f_i : \mathbb{R} \to \mathbb{R}, \; L \in \mathbb{N} \setminus \{0,1\}, \; t_i \in \mathcal{N},$$

where $\mathcal{N} \subset \mathbb{R}$ is a lattice. (Systems of the form where $L \in \mathbb{N}$ and the translations are rational numbers are example of such systems.) Following [4] we call IFSs of this form *special family of finite type*—because in particular these IFSs satisfies the finite type condition (for more details see the cited book). It can be shown that rational projections of random grid aligned sets are of this above form.

3.0.1. *Dimension.* So far we have seen some results about the dimension of :

(1) grid aligned sets, like Mandelbrot percolation, where the a.s. Hausdorff dimension and the upper and lower box counting dimension agrees, and is equal to the similarity dimension $\alpha = \log_K(K^d p)$ (see (2.2) above).
(2) projections onto coordinate axes, such as in Theorem 2.1 where the almost sure dimension is given by the formula $\frac{\inf\limits_{0 \le s \le 1} \left[\log \sum\limits_{i=0}^{K-1} m_i^s \right]}{\log K}$
(3) more general coin tossing self-similar IFSs, which are special cases of so called ∞-variable random graph directed systems (RGDS) which were (among other things) considered in [44] by Troscheit. In this paper he proves equality of the Hausdorff and the upper and lower box dimension attractors of RGDSs under mild conditions which are satisfied for coin-tossing self similar IFSs consisting of functions with at least two different fixed points. Hence, by [44, Theorem 3.5],

$$\dim_{\mathrm{H}} \Lambda_{\mathcal{F}}(p) = \dim_{\mathrm{B}} \Lambda_{\mathcal{F}}(p). \tag{3.1}$$

However, exact value of the dimension is generally not known.

3.0.2. *Running example.* In what follows we will present results about special coin tossing systems of finite type. We show these through our running example, the random $0-1-3$ set with contraction ratio $1/2$. The $0-1-3$ set with contraction ratio $1/2$ is the attractor of the IFS

$$\left\{ S_i(x) = \frac{1}{2}x + t_i \right\}_{i=1}^{3}, \quad t_i = 0, 1, 3. \tag{3.2}$$

For an illustration of the first level cylinders see Figure 4. In what follows we will examine a similar partition of $[0,1]$, the parameter interval of p as it appears in Figure 2. These properties depend on the behaviour of the so-called expectation matrices, which are the multidimensional analogues of the m_i's introduced in the previous section. The methods used in the proofs resemble the ones used in the case of projections to the coordinate axes (or more generally systems satisfying the OSC), with some complexity introduced by the presence of matrices. We start with the setup, including the construction of the expectation matrices.

3.1. Constructing the expectation matrices.

To construct expectation matrices corresponding to the $0-1-3$ set, we construct matrices corresponding to the deterministic system.

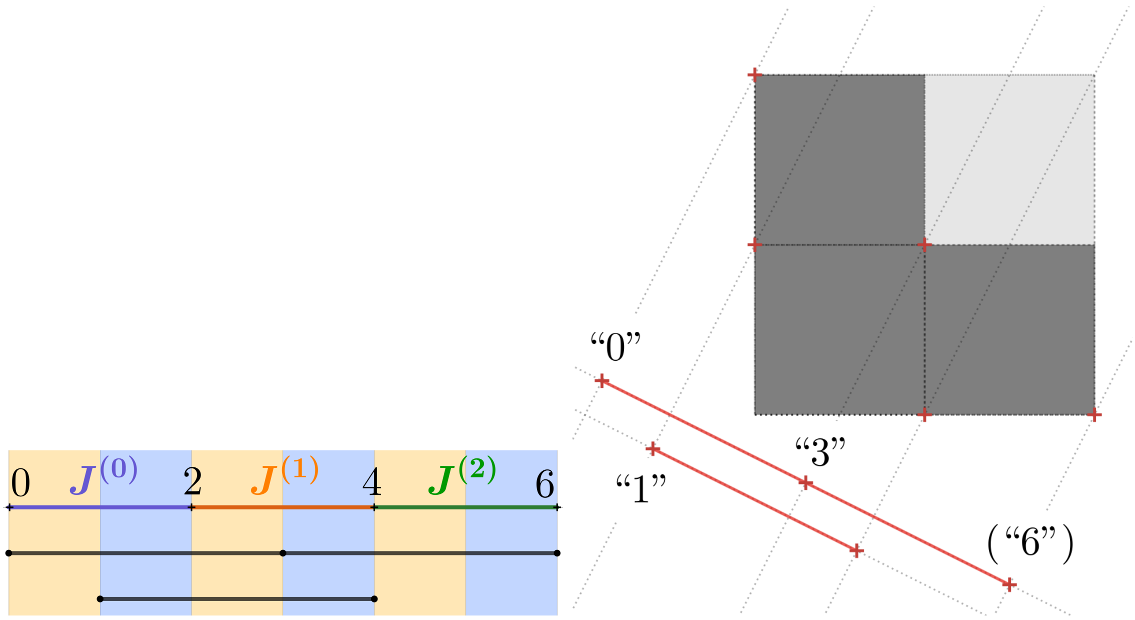


FIGURE 3. 0-1-3

3.1.1. *Matrices for the deterministic systems.* In this case the contraction ratio is the reciprocal of $L = 2$, the number of functions is $M = 3$. Consider the dyadic intervals

$$\mathcal{D}_k := \left\{ \left[(i-1)2^{-k}, i2^{-k} \right] : i \in \mathbb{Z} \right\}, \quad k \in \{-1, 0, 1, 2, \dots\}. \tag{3.3}$$

Consider $J^{(0)} = [0,2], J^{(1)} = [2,4], J^{(2)} = [4,6] \in \mathcal{D}_{-1}$, called *basic intervals*. The motivation for the name is coming from the observation that for all $\mathbf{i} \in [M]^n$ we have $S_{\mathbf{i}}(J^{(k)}) \in \mathcal{D}_{1-n}$. This helps keeping track of the number of cylinders intersecting the dyadic subintervals $J_\theta^{(k)}$ of $J^{(k)}$. The expectation matrices will be used for this purpose. For $k \in [3]$, and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \{0,1\}^m$ we denote

$$J_{\boldsymbol{\theta}}^{(k)} := \left[a_k + \sum_{\ell=1}^{n} \theta_\ell 2^{-(\ell-1)}, b_k + \sum_{\ell=1}^{m} \theta_\ell 2^{-(\ell-1)} + 2^{-(m-1)} \right],$$

where the left and right endpoints of the interval $J^{(k)}$ are a_k and b_k respectively.

In general the number of matrices is L, in the case of the $0-1-3$ example $L = 2$, all of which have N rows and columns, where N is the number of basic intervals, in this example $N = 3$. For $\theta \in [L]$ and $i, k \in [N]$:

$$\mathbf{B}_\theta(i,k) = \# \left\{ \ell \in [M] : S_\ell(J^{(k)}) = J_\theta^{(i)} \right\}, \text{ and} \tag{3.4}$$

$$\mathbf{B}_{\boldsymbol{\theta}}(i,k) = \left(\mathbf{B}_{\theta_1} \cdots \mathbf{B}_{\theta_n} \right)(i,k) = \# \left\{ (\ell_1, \dots, \ell_n) \in [M]^n : S_{\ell_1 \dots \ell_n}(J^{(k)}) = J_{\boldsymbol{\theta}}^{(i)} \right\}. \tag{3.5}$$

For example the first row of $\mathbf{B}_1$ constructed as follows: The first row describes the cylinders intersecting $J_1^{(0)} = [1,2]$. One can verify that $S_0(J^{(1)}) = S_1(J^{(0)}) = J_1^{(0)}$, meaning that $\mathbf{B}_1(0,0) = 1$ (from S_1), and $\mathbf{B}_1(0,1) = 1$ (from S_0). This can be also read from Figure 4, for constructing $\mathbf{B}_1$ we inspect the blue stripes, for the first row we consider the blue stripe intersecting $J^{(0)}$ and for the i-th element of this row we count the number of maps from the IFS that maps $J^{(i)}$ to this stripe. Similarly, we can construct the remaining part of $\mathbf{B}_1$ and also $\mathbf{B}_0$. Altogether

$$\mathbf{B}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.1.2. *Expectation matrices.* The expectation matrices are

$$\mathbf{M}_\theta(i,k) = \mathbb{E}\left(\#\left\{\ell \in [M] \cap \mathcal{E}_1 : S_\ell(J^{(k)}) = J_\theta^{(i)}\right\}\right). \tag{3.6}$$

Since the probability of retaining a given cylinder is p, we have

$$\mathbf{M}_i = p \cdot \mathbf{B}_i. \tag{3.7}$$

Throughout this paper we will use the notation $p \cdot \mathbf{B}_i$ for emphasizing the dependence on p.

3.2. **Results applied to the random 0-1-3 set with contraction ratio $1/2$.**

3.2.1. *Positivity of Lebesgue measure.* We will use Theorem 3.5. of the paper [33], to give a $p_{\mathcal{L}}$ so that for $p > p_{\mathcal{L}}$ the random set has positive Lebesgue measure almost surely conditioned on non-extinction. The theorem states that if the expectation matrices satisfy some property (we clarify in (3.2.1)), then given $p > \mathrm{e}^{-\lambda}$, the set has positive Lebesgue measure almost surely, where λ is the Lyapunov exponent corresponding to the matrices $\mathbf{B}_0$ and $\mathbf{B}_1$ with respect to the uniform measure $\{1/2, 1/2\}^{\mathbb{N}}$ on $\{0,1\}^{\mathbb{N}}$. Conversely, according to the above-mentioned Theorem 3.5 whenever $p < p_{\mathcal{L}}$ the Lebesgue measure is 0 almost surely. The property mentioned above the matrices has to satisfy is called "goodness" in the paper, which means in this particular case, that

- every matrix has a positive element in every row and every column, and
- there exists a product of the matrices which is strictly positive (i.e. every element of it is positive).

The first condition is clearly satisfied, and for the second, consider

$$\mathbf{B}_0\mathbf{B}_1\mathbf{B}_0\mathbf{B}_0 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}. \tag{3.8}$$

Remark 3.1. In [32] we give more checkable conditions which depend only on the column sums of the expectation matrices (see Theorem 1.11). However, the theorem can only be applied in the case when each matrix has a strictly positive row, which is not the case for the matrices corresponding to $0-1-3$ set. The theorem is often applicable when the IFS contains many distinct functions. In that paper we consider the projection of the random Menger sponge to the diagonal of the unit cube, which satisfy this condition, see [32, Section 2.1].

Remark 3.2. The proof of the theorem is—as is typical—based on a branching process argument. The special type of branching process considered in the paper is called multitype branching process in random environment. The survival of this process is equivalent to Lebesgue almost every point of the set surviving with positive probability. By self-similarity, this implies the almost sure statement (the base of this standard argument is explained in [32, Lemma 3.9]).

3.2.2. *Interior points.* The existence of interior points in the set can also be phrased in terms of the expectation matrices. About interior points we know the following:

(1) If there exists a product consisting of $p \cdot \mathbf{B}_0$ and $p \cdot \mathbf{B}_1$ which has spectral radius less than 1, then the set has empty interior almost surely. This follows from [33, Proposition 6.2], combined with the fact that the lower spectral radius given a set of matrices is lower bounded by the n-th root of the spectral radius of any n-term matrix product.
(2) By [32, Theorem 1.9], if there exists a product of the matrices $\mathbf{B}_0$ and $\mathbf{B}_1$ with a strictly positive row, and moreover every column sum of $p \cdot \mathbf{B}_0$ and $p \cdot \mathbf{B}_1$ is greater than 1, then the random set (with parameter p) has interior point almost surely conditioned on non-extinction. (In our example, the $0-1-3$ set, the first condition is satisfied, but the second is only satisfied when $p = 1$ and therefore we cannot apply this theorem. In the above cited paper we again consider the Menger sponge, where this condition is usable and moreover gives the sharp bound.)
(3) By the at first glance technical [33, Proposition 7.2], if we can find a set of non-zero vectors with integer coordinates $\mathcal{U} = \{u_1, \dots, u_m\}$ such that
   (a) there exists a coordinate i and a finite word $\theta \in [L]^s$ and a $u \in \mathcal{U}$ so that $e_i^T \mathbf{B}_\theta \geq u$, and
   (b) there exists a $\gamma' > 1$ and a level S' such that for all $\underline{\theta} \in [L]^{S'}$ there exists a non-negative, $|\mathcal{U}| = m \times m$ matrix $\mathbf{A}_{\underline{\theta}}$ with all row sums greater than γ' (i.e. for all $i \in [|\mathcal{U}|]$ $\sum_{k \in [|\mathcal{U}|]} \mathbf{A}_{\underline{\theta}}(i,k) > \gamma' > 1$). Assume that for this $\mathbf{A}_{\underline{\theta}}$

$$\mathbf{U}\mathbf{M}_{\underline{\theta}} \geq \mathbf{A}_{\underline{\theta}}\mathbf{U}, \tag{3.9}$$

   where $\mathbf{U}$ is the $|\mathcal{U}| \times N$ matrix having row vectors $\mathbf{u}_i^T$ for $i = 1, \dots, m$.

As we noted, out of the above only the first and the third can be applied, however these won't give a sharp bound on the critical probability corresponding to the existence of interior point. For a lower bound on the critical probability we used numerical estimations of the spectral radius of the products up to level 4, this gave $\underline{\rho} \leq \lim_{k \to \inf} \|M^k\|^{1/4k} \sim 1.41954$, this gives the lower bound $p = 0.704$. For an upper bound given by the 3-rd point, we used $\mathfrak{U} = \{(1,0,1),(1,1,0),(0,1,1)\}$, and we estimated the optimal γ at level 20, which gave us the following result: for $p > 463^{-1/20} \sim 0.7357$, $\Lambda_{\mathcal{S}}(p)$ contains an interval almost surely conditioned on non-extinction.

Remark 3.3. Truly, the third point might seem technical and ungrounded, but it's just the natural generalization of the second point. This is explained in more detail in the paper, however we try to briefly explain it here. The second point, (2) is about how it is necessary that expected number of type-triplets (containing the interval type $J^{(0)}$, $J^{(1)}$ and $J^{(2)}$ simultaneously) coming from one type is at least one. The third generalizes this in a way that we can choose a set of "constellation" of types (the constellations are the u_i vectors of the set $\mathfrak{U}$) so that the expected number of all the constellations is at least 1. Also for the set $\mathcal{U} = \{(1, \dots, 1)\}$ this is precisely (2).

3.3. **Some possible extensions.**

(1) Higher dimensional systems. *The same frameworks are usable in higher dimensional systems as well (see Example 3.5 below). However, in higher dimensional systems it is natural that the matrices are sparser, which can make the theory be non-applicable (in the worst case scenario the matrices are not allowable).*

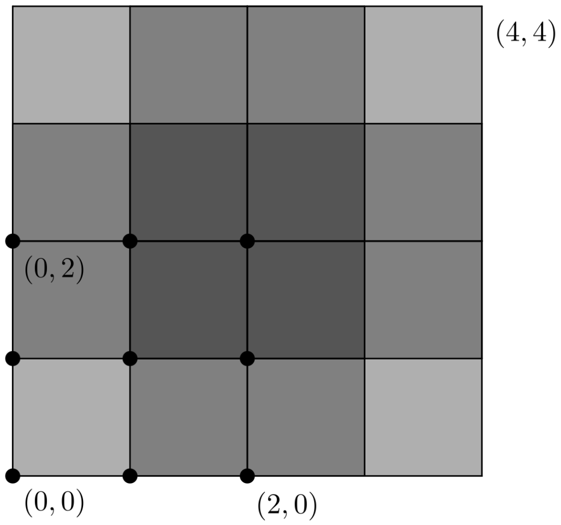


FIGURE 4. 0-1-3

(2) Non-homogeneous probabilities. *It is possible to consider systems, where instead of choosing a probability* p*, we choose a probability vector* $\underline{p} = (p_0, \dots, p_{M-1})$ *and at each retained cylinders at each level we retain or discard the i-th cylinder with probability* p_i *and* $1 - p_i$ *respectively.*
(3) Irrational directions. *It is possible to consider d-dimensional systems that are projections of higher-dimensional sets to non-rational directions, or more generally where the translations are all irrational.* In this case the expectation matrix is impossible to build, in the sense that there is no finitely many "type". Almost every translation type statements can sometimes be handled using the generalized projection scheme.
(4) Non-homogeneous contractions. The treatment of finite type IFS's suggests that there is a way to extend the result to the case when: there exists a θ PV number, so that every contraction is of the form θ^{-n_i} for $n_i \in \mathbb{N}$ and all the translations are in $r_1\mathbb{Z}[\theta] \times \cdots \times r_k\mathbb{Z}[\theta]$. However, the matrices are not the finite type matrices. They describe the system in a more direct way, so the proof does not generalize to this case.
(5) If all column sums agree, the Lyapunov exponent and the lower spectral radius are the same. Thus one might wonder, if *not* all the column sums agree, whether there exists a parameter interval of p (say (p_1, p_2), $0 < p_1 < p_2 \leq 1$) so that for $p \in (p_1, p_2)$ the set has positive Lebesgue measure but non-empty interior (almost surely conditioned on non-extinction). As it is explained in [33] this depends on the behaviour of the pressure function on the non-positive half line. If it is a straight line the above-mentioned parameter interval *does not exist*, however if it is not, then this interesting parameter interval does exists.

Remark 3.4. We repeat here the 2-dimensional example from [33]

Example 3.5 (Overlapping Mandelbrot percolation)*.* The following is one of the simplest 2-dimensional examples and for the first level cylinders see Figure 4.

$$\mathcal{S} = \left\{ \frac{\mathbf{x}}{2} + \mathbf{t}_i \right\}_{i=1}^{9}, \tag{3.10}$$

where $\mathbf{t}_i$ runs through the set $\{0, 1, 2\}^2$.

Lemma 3.6. *In the above overlapping Mandelbrot percolation example we have:*

- Λ_p *contains a ball almost surely conditioned on non extinction iff* $p = 1$.
- *When* $p > 0.993$ *then by* [17] *the set contains a curve which connects the left and right side of the unit square with positive probability.*
- *When* $p > 0.7712$ *then the set has positive two dimensional Lebesgue measure almost surely conditioned on non-extinction.*

Note, that in this case $N = 4$, and the corresponding matrices are:

$$\mathbf{B}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \mathbf{B}_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \mathbf{B}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4. Slices and Projection in arbitrary directions

In this section we consider homogeneous Mandelbrot percolation sets in dimension 2. Throughout this section we use the shorthand notation $\Lambda (:= \Lambda(K, p))$, for the Mandelbrot percolation with parameters K and p.

Recall that $\#\mathcal{E}_n$ (the number of retained level n squares in the Mandelbrot percolation) is a Galton–Watson branching process with offspring distribution $Bin(K^2, p)$. In this section we use the notation Λ_n for the n-th approximation of the Mandelbrot percolation set Λ;

$$\Lambda_n := \bigcup_{\mathtt{i} \in \mathcal{E}_n} Q_{\mathtt{i}},$$

where $\{Q_{\mathtt{i}}\}_{\mathtt{i} \in [M]^n}$ is the natural partition into K-adic cubes at level n.

From the general theory of Galton–Watson processes (see [2]), it is well known that

$$W := \lim_{n \to \infty} \frac{\#\mathcal{E}_n}{(K^2 \cdot p)^n}, \tag{4.1}$$

and $W > 0$ almost surely conditioned on $\Lambda \neq \emptyset$ and $\mathbb{E}[W] = 1$. It follows from (4.1) above that

$$\lim_{n \to \infty} \frac{\#\mathcal{E}_n \cdot K^{-2n}}{p^n \cdot W} = 1. \tag{4.2}$$

The natural measure on Λ is defined as the weak limit

$$\mu := \lim_{n \to \infty} \frac{\mathcal{L}eb_2|_{\Lambda_n}}{\mathcal{L}eb_2(\Lambda_n)} = \lim_{n \to \infty} \frac{\mathcal{L}eb|_{\Lambda_n}}{\#\mathcal{E}_n \cdot K^{-2n}} = \lim_{n \to \infty} \frac{\mathcal{L}eb|_{\Lambda_n}}{p^n \cdot W}. \tag{4.3}$$

It was proved by Mauldin and Williams [29] that this limit exists. As was later noticed by Peres and Rams [36] to study the natural measure μ, it is convenient to consider the measure

$$\widetilde{\mu} := W \cdot \mu = \lim_{n \to \infty} \underbrace{\frac{\mathcal{L}eb|_{\Lambda_n}}{p^n}}_{=:\widetilde{\mu}_n}$$

(recall, that $W > 0$ a.s. conditioned on non-extinction), since this sequence of measures $\{\widetilde{\mu}_n\}_{n=1}^{\infty}$, is a martingale. That is:

$$\widetilde{\mu}_n(H) = \mathbb{E}\left[\widetilde{\mu}_{n+1}(H) | \mathcal{E}_n\right] \tag{4.4}$$

for every Borel set H. See [36].

4.1. **Projections of the homogeneous Mandelbrot percolation sets.** We write proj_θ for the orthogonal projection to the line which has angle θ with the positive part of the x-axis. Let $t \in \mathbb{R}^2$. The radial and co-radial projections ($\mathrm{Proj_t}$ and $\mathrm{CProj_t}$ respectively) of center t are defined on Figure 5.

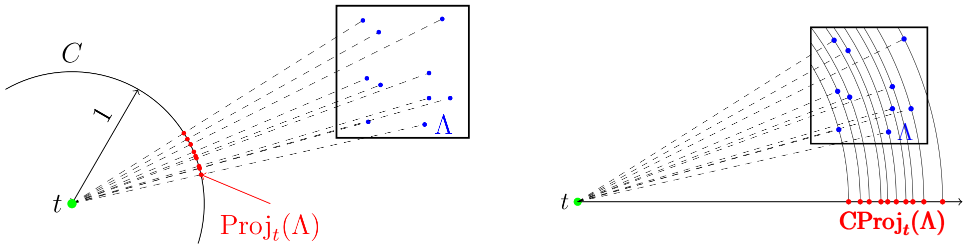


FIGURE 5. The radial and co-radial orojections

The following theorem was proved in [40].

Theorem 4.1 (Rams, Simon)**.** *Given $p > 1/K$, the following statements hold almost surely conditioned on $\Lambda \neq \emptyset$:*

$$\forall \theta \in [0, \pi], \ \mathrm{proj}_\theta(\Lambda) \textit{ contains an interval} . \tag{4.5}$$

Further,

$$\forall t \in \mathbb{R}^2, \ \mathrm{Proj}_t(\Lambda) \textit{ and } \mathrm{CProj}_t(\Lambda) \textit{ also contains an interval} . \tag{4.6}$$

The assumption $p > \frac{1}{K}$ is equivalent to $\dim_{\mathrm{H}} \Lambda > 1$ a.s. (conditioned on non-extinction). If this condition does not hold then every projection of Λ has empty interior. We can popularize (4.5) as follows: In the next section we will introduce p_c, a critical probability so that for $p < p_c$ then Λ is totally disconnected almost surely, whereas if $p \geq p_c$ then it contains a connected component connecting the left and right side of the unit square. It was proven by Chayes, Chayes and Durrett that $p_c < 1$. If $p > \frac{1}{K}$ but p is less than a critical probability then Λ is a random dust but every projection of Λ contains an interval. That is the shadow of Λ at every time contains an interval as depicted in Figure 6.

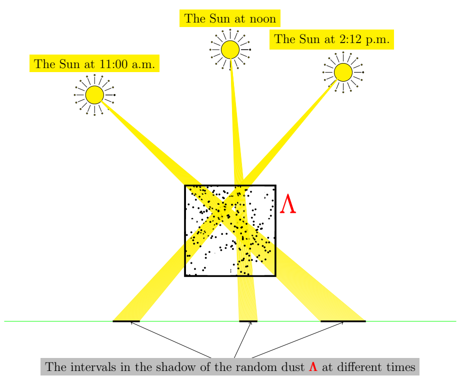


FIGURE 6. Each projection of the random dust Λ contains some intervals

The following theorem is the dimension counterpart of the previous one, and it was proved in [38].

Theorem 4.2. *If $\frac{1}{K^2} = \frac{1}{M} < p \leq \frac{1}{K}$ then for almost all realizations of Λ (conditioned on $\Lambda \neq \emptyset$), and for all straight lines ℓ: There exists a constant C such that the number of level n squares having non-empty intersection with Λ is at most $C \cdot n$.*

On the other hand, almost surely for n big enough, we can find some line of $45°$ angle which intersects $\texttt{const} \cdot n$ level-n squares.

It was proved by Simon and Vágó in [42] that the higher dimensional analogues of the previous theorems also hold in the same way.

Peres and Rams [36] proved the following theorem concerning the absolute continuity of the natural measure:

Theorem 4.3 (Peres, Rams)**.** *Assume that $Kp > 1$. Then conditioned on non-extinction, for almost all realization the following holds: for all θ the projected measure $\mu_\theta := (\mathrm{proj}_\theta)_* \mu$ of the natural measure μ is absolute continuous. Moreover, if $\theta \notin \{0, \pi/2\}$ then the density is Hölder continuous. For the vertical and horizontal directions the density in not defined at the K-adic points but otherwise it is Hölder continuous for a specially chosen metric.*

The most important tool of the proof of this theorem was the martingale property (4.4) of the modified version $\widetilde{\mu}$ of the natural measure μ. Based on the generalization of this martingale property, Shmerkin and Suomala presented much more general and interesting results in [41], which exceeds the scope of this survey.

4.2. **Slices of homogeneous Mandelbrot percolations.** Recall that proj_θ is the orthogonal projection onto the line ℓ_θ, which was defined as the angle-θ line. Let $x \in \ell_\theta$. Then for an arbitrary set $T \subset \mathbb{R}^2$ we define $T(\theta, x)$ the (θ, x)-slice of T by

$$T(\theta, x) := T \cap (\mathrm{proj}_\theta)^{-1}(x) = (x + \ell_{\theta^\perp}) \cap T.$$

The following result is [41, Corollary 12.10]:

Theorem 4.4. *Assume that $Kp > 1$ then for almost all realizations for all $\theta \notin \left\{0, \frac{\pi}{2}\right\}$ there exists a non-empty interval $U_\theta \subset \ell_\theta$ such that for all $x \in U_\theta$:*

$$\dim_{\mathrm{H}} \Lambda(\theta, x) = \frac{\log\left(K^2 p\right)}{\log K} - 1 = \dim_{\mathrm{H}} \Lambda - 1 = \frac{\log(Kp)}{\log K}. \tag{4.7}$$

4.3. **Interval in the images of homogeneous Mandelbrot percolation sets.**

Definition 4.5. We say that $f : [0,1]^2 \to \mathbb{R}$ is a strictly monotonic smooth function if $f \in \mathcal{C}^2[0,1]$ and $f'_x \neq 0,\ f'_y \neq 0$.

Theorem 4.6 (Rams, Simon)**.** *If $Kp > 1$ then for every strictly monotonic smooth function f, $f(\Lambda)$ contains an interval, almost surely conditioned on non-extinction.*

Examples: If $Kp > 1$ then

- $\{x + y : (x, y) \in \Lambda\} \supset$ interval .
- $\{x \cdot y : (x, y) \in \Lambda\} \supset$ interval .

The following theorem ([38, Theorem 17]) was already mentioned in the introduction, however because of the connection we repeat it.

Theorem 4.7 (Rams, Simon)**.** *Let $d \geq 2$ and let E^i be independent copies of one dimensional Mandelbrot percolations with the same parameter K but possibly different probabilities: Λ^1_{K,p_i} for $i = 1, \dots, d$. Assume that $\prod_{i=1}^{d} p_i > K^{-d+1}$. Then for every*

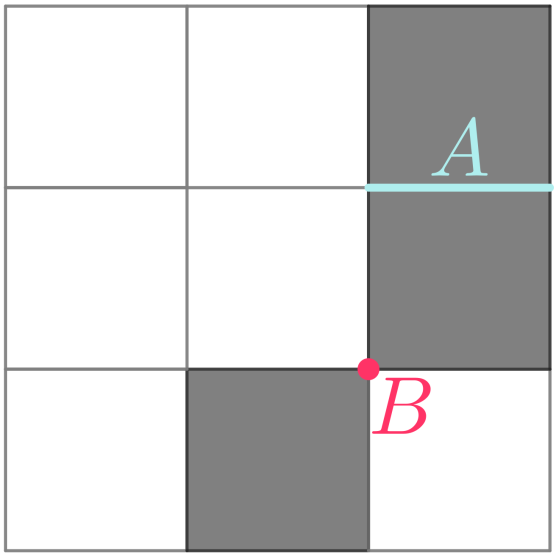


FIGURE 7. Here in this figure the dark squares are retained; the connection denoted by B is a vertex connection and the connection denoted by A is a side connection.

$\mathbf{b} = (b_1, \dots, b_d) \in \mathbb{R}^d$, with $b_i \neq 0$ for all $i = 1, \dots, d$ the sum set $E_{\mathbf{b}}^{\text{sum}} := \sum_{i=1}^{d} b_i E^i$ contains an interval.

5. Connectivity properties

In this subsection we present some of the history of presence of connected components, connecting the left and the right side of the unit square in Mandelbrot percolation fractals.

5.1. **Notions and notations.** By percolation we mean that with positive probability there exists a connected component in the Mandelbrot percolation set connecting the left and the right sides of the unit square. The event of percolation, denoted by B^{LR}, is the intersection of the events

$$B_n^{LR} := \{\text{There exists a side connected component connecting the left and the right wall of } [0,1]^2 \text{ made of retained level n squares}\},$$

where by "side connected" we mean that the squares share a side and not just a vertex; see Figure 7. This at first might seem counterintuitive, however those connected components that relies on a vertex connection eventually disconnects as it is explained by Lincoln Chayes in the nice survey [9, Section 4] on percolation. In what follows we will briefly summarize the results concerning the existence of a critical probability p_c such that for $p_\emptyset < p < p_c$ the Mandelbrot percolation set is dust-like, i.e. every connected component is a single point almost surely and for $p \geq p_c$ the Mandelbrot percolation set contains a left-right crossing with positive probability (and hence, by statistical self-similarity also the set contains non-trivial connected components almost surely on non-extinction).

Some of the results regarding the existence and properties of percolation are as follows (in more or less historical order):

- Chayes, Chayes and Durrett [8] (partially based on the ideas of Mandelbrot) showed that this critical probability exists, and surprisingly it is smaller than one and further that the percolation probability function is right continuous and discontinuous at the critical probability.
- The above result concerning discontinuity at the critical probability was later also established by Dekking and Meester in [15], where the authors considered the phase transitions of the Mandelbrot percolation set (and also the random Sierpinski carpet). The proof here for discontinuity is simpler than the above, and works for the Sierpiński carpet as well. (It is also proved in the above mentioned survey [9]).
- In 1992 Meester [30] proved that almost surely if there is a connected component that connects the left and right sides, then almost surely there is an arc connected component with the same properties. He also proved that if $p > p_c$, then almost surely conditioned on non-extinction, the number of retained components is countably infinite whereas the set of isolated points is uncountable.
- Recently Falconer and T. Feng considered (see [21]) the self-affine analogue of the Mandelbrot percolation (i.e. percolation on an $N \times M$ grid). They proved that the critical probabilities for the horizontal and vertical crossings are the same and also considered some properties mentioned in the previous point for this self affine version.
- About the regularity of the percolation path:
  - In 1996 Chayes ([10]) proved that every curve contained in the set has lower box dimension at least $1 + \zeta$ for some $0 < \zeta(M, p)$ uniform constant. To establish this result, he proved that with probability one there cannot exist a directed curve (that is, moving only left and up), that connects the left and the right sides and that the existence of such path is required for ζ to be equal to 0.
  - In 1998 in a paper and his PhD thesis ([34], [35]) Orzechowski further investigated the question of box-dimension of the percolation curves, and gave a sharper lower bound on the lower box dimension of any percolationg curve contained in the percolation set. He investigated the a.s. existence of a curve with upper box dimension less than some $\beta(M, p)<2$.
  - In 2013 Broman, Camia, Joosten and Meester ([6]) proved that for Λ_d (which denotes the "dust-like" component of Λ, namely in which every connected component is a single point), and $\Lambda_c = \Lambda \setminus \Lambda_d$ and for $p \geq p_c$ we have the following:
    * $\dim_B(\Lambda_c) = \dim_H(\Lambda)$, however:
    * conditioned on non extinction $\dim_H(\Lambda_c)$ is almost surely a constant satisfying $\dim_H(\Lambda_c) < \dim_H(\Lambda)$, and in particular $\dim_H(\Lambda) = \dim_H(\Lambda_d)$.
  - Last but not least in 2021 Buczolich, Järvenpää, Järvenpää, Keleti and Pöyhtäri in [7] proved that there exists $0 < \alpha_0 < 1$ (depending on the parameters) such that the fractal percolation is almost surely purely α-unrectifiable for all $\alpha > \alpha_0$.

5.2. **Estimations of p_c.** In what follows $p_c(K)$ denote the critical probaility in case of the Mandebrot percolation with parameter K. We will not discuss in more detail, however estimations of p_c were also considered by Klatt and Winter in 2020 and 2024 in [27], [28].

- Lower bounds:
  (1) The first estimate of p_c is in [8] and the idea is as follows: For percolation to occur, every vertical line $k \times [0,1]$ must be crossed by a percolating path, from which it follows that there is a pair of retained squares S_1^n, S_2^n sharing an edge which are on opposite sides of the line. This forms a branching process with expected number for children: $M \cdot p^2$, hence $p_c > \frac{1}{\sqrt{M}}$.
  (2) In 2001 White ([47]) improved this to $p_c(2) \geq 0.8107$ using computer-aided techniques.
  (3) Eventually in 2015 Henk Don (also using computer-aided techniques) improved this to $p_c(2) > 0.881$. Further he showed that $p_c(3) > 0.784$; see [18] or his PhD thesis [17].
- Upper bounds:
  (1) A way to estimate p_c from above is to estimate the probability p such that for $p > p$ we have that the probability that at least $M^2 - 1$ square is retained is positive (this is also the idea of the first proof of $p_c < 1$). This was improved by Dekking and Meester in [15] (for the random Sierpiński carpet, which is also a good bound for the Mandelbrot percolation) by considering more complicated constructions than just "retain 8 squares", to estimate the probability.
  (2) We also mention that van der Wall improved this in [46] to $p_c(4) < 0.998$ and $p_c(2) < 1 - 10^{-12}$.
  (3) Don in [18] (or see his PhD thesis [17]) proved that $p_c(2) < 0.993$, $p_c(3) < 0.940$ and $p_c(4) < 0.972$.

Appendices

A. Mandelbrot cascades

A.0.1. *Construction.* In this section we closely follow Falconer and Jin [23]. A random multiplicative cascade is essentially a random measure on $[M]^{\mathbb{N}}$ constructed in a statistically self-similar manner. Mandelbrot cascades are in some sense much more general versions of coin tossing systems and the Mandelbrot percolation set; in a special case cascade measure (described below) and its natural projection to $\mathbb{R}^d$, then we get back the rescaled natural measure on the Mandelbrot percolation.

The natural projection $\pi : [M]^{\mathbb{N}} \to \mathbb{R}^d$ is

$$\pi(i) := \lim_{n\to\infty} f_{i|n}(x), \tag{A.1}$$

for some fixed $x \in \mathbb{R}^d$. In what follows we describe the general system for reference tightly following Falconer and Jin in [23]. Then we provide some descriptive examples.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let

$$W = (W_0, \dots, W_{M-1}) \in [0,\infty)^m$$

be a random vector defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\sum_{i\in[M]} \mathbb{E}(W_i) = 1$. Let $\{W^{[\mathtt{i}]} : \mathtt{i} \in [M]^*\}$ be a sequence of independent and identically distributed random vectors having the same law as W (so we assign a random vector for each vertex of a full M-ary tree). For a given $\mathtt{i} \in [M]^*, n \geqslant 1$ and $\mathtt{j} = j_1 \cdots j_n \in [M]^n$, define

$$Q_{\mathtt{j}}^{[\mathtt{i}]} = W_{j_1}^{[i]} W_{j_2}^{[ij_1]} \cdots W_{j_n}^{[ij_1\cdots j_{n-1}]},$$

describing the random (vector valued) variables on branch $\mathtt{j}$ of the node $\mathtt{i}$ (we inspect the nodes corresponding to the word $\mathtt{j}$ in the tree rooted in node $\mathtt{i}$).

For $\mathtt{i} \in [M]^*$ and $n \geqslant 1$ define $Y_n^{[\mathtt{i}]} = \sum_{\mathtt{j} \in [M]^n} Q_{\mathtt{j}}^{[\mathtt{i}]}$, which describes the tree rooted in $\mathtt{i}$ of depth n. By definition, $\left\{Y_n^{[\mathtt{i}]}\right\}_{n \geqslant 1}$ is a non-negative martingale. Assume that

$$\text{(a0) } \mathbb{P}\left(\#\left\{i \in [M] : W_i > 0\right\} > 1\right) > 0,$$

$$\text{(a1) There exists } p > 1 \text{ such that } \sum_{i=1}^{m} \mathbb{E}\left(W_i^p\right) < 1.$$

Then $Y_n^{[\mathtt{i}]}$ converges a.s. to a non-trivial limit $Y^{[\mathtt{i}]}$. $\mathbb{E}\left(Y^{[\mathtt{i}]}\right) = 1$, $Y^{[\mathtt{i}]}, \mathtt{i} \in [M]^*$ have the same law as $Y = Y^{[\emptyset]}$. Since $[M]^*$ is countable, $Y^{[\mathtt{i}]}$ is well defined for all $\mathtt{i} \in [M]^*$ simultaneously. Then for each $\mathtt{i} \in [M]^*$, we may define a random measure $\mu^{[\mathtt{i}]}$ on $[M]^{\mathbb{N}}$ by

$$\mu^{[\mathtt{i}]}([\mathtt{j}]) = Q_{\mathtt{j}}^{[\mathtt{i}]} \cdot Y^{[\mathtt{ij}]}, \quad \mathtt{j} \in [M]^*. \tag{A.2}$$

The measure $\mu^{[\mathtt{i}]}$ is called the random multiplicative cascade measure generated by the sequence $\left\{W^{[\mathtt{ij}]} : \mathtt{j} \in [M]^*\right\}$. The measure in the root $\mu^{[\emptyset]} =: \mu$ will be in the center of our interest.

In what follows we will need the countable product space

$$(\Omega^*, \mathcal{F}^*, \mathbb{P}^*) = \prod_{\mathtt{i} \in [M]^*} (\Omega_{\mathtt{i}}, \mathcal{F}_{\mathtt{i}}, \mathbb{P}_{\mathtt{i}}), \tag{A.3}$$

$(\Omega_{\mathtt{i}}, \mathcal{F}_{\mathtt{i}}, \mathbb{P}_{\mathtt{i}}) = (\Omega, \mathcal{F}, \mathbb{P})$. For the probability conditional on μ being non-trivial we write

$$\mathbb{P}_*(A) = \frac{\mathbb{P}^*(A \cap \{\mu(\Sigma) > 0\})}{\mathbb{P}^*(\{\mu(\Sigma) > 0\})} \tag{A.4}$$

for $A \subset \mathcal{F}^*$ and

$$\overline{\mu}(A, \omega) = \chi_{\mu(\Sigma) \neq 0}(\omega) \frac{\mu(A)}{\mu(\Sigma)}. \tag{A.5}$$

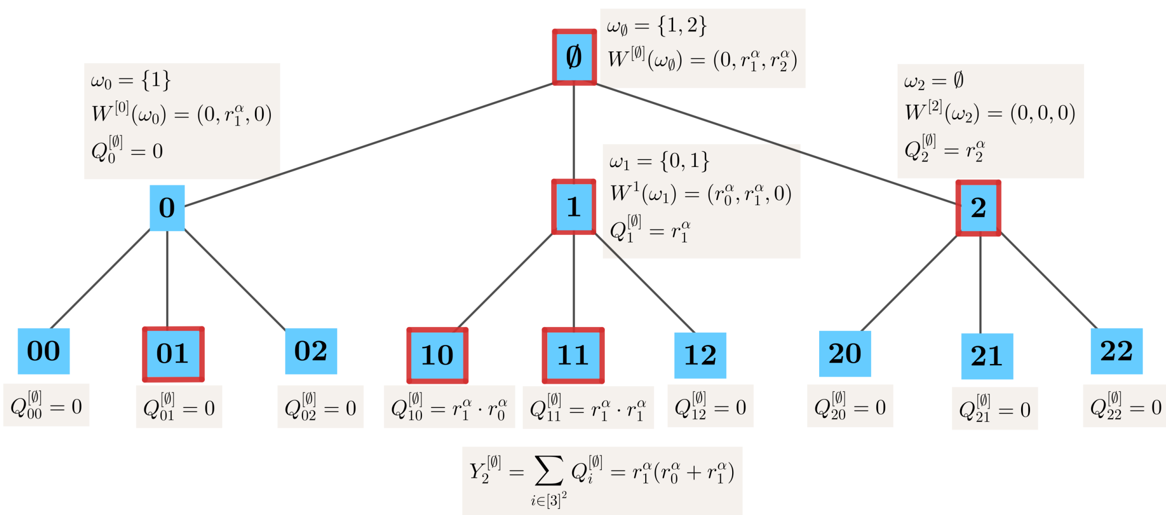


FIGURE 8. An example realization for the cascade construction in the case of coin tossing systems.

.

A.0.2. *Special cases.*

Example A.1 (Coin tossing systems, Section 6 of [23]). We will continue working with the IFS (1.1). Fix a probability $p \in [0,1]$. A particular realization of the below is represented in Figure 8. Consider the probabilty space $(\widehat{\Omega}, \mathcal{F}, \mathbb{P})$, where $\widehat{\Omega} = 2^{[M]}$, $\mathcal{F} = 2^{\widehat{\Omega}}$. For $\omega \in \widehat{\Omega}$ (ω is coding a subset of $[M]$) we define

$$\mathbb{P}(\omega) = p^{\#\omega}(1-p)^{M-\#\omega},$$

so the probability of a particular subset of $[M]$ equals to the probability that we get the same subset by tossing a coin independendently for each M element of the set $[M]$, which gives 1 with probability p and 0 with probability $1-p$. Let $W(\omega) = (W_0(\omega), \dots, W_{M-1}(\omega))$ be a random vector so that $W_i(\omega) = \chi_{i\in\omega} r_i^{\alpha}$, where α is chosen so that

$$p \sum_{i=1}^{M} r_i^{\alpha} = 1,$$

where recall that r_i is the contracting ratio corresponding to the i-th map of the IFS. The reason for this choice is that in this way $\mathbb{E}(\sum_{i=1}^{M} W_i) = 1$. Let $\{W^{[\mathtt{i}]}\}_{\mathtt{i}\in[\mathtt{M}]^*}$ be i.i.d. with $W^{[\mathtt{i}]} \overset{d}{=} W$. In this case

$$Q_j^{[\mathtt{i}]} = W_{j_1}^{[\mathtt{i}]} W_{j_2}^{[\mathtt{i}j_1]} \cdots W_{j_n}^{[\mathtt{i}j_1\cdots j_{n-1}]} = \begin{cases} 0, & \text{if any of } W_{j_m}^{[\mathtt{i}j_1\dots j_{m-1}]} = 0, \\ (r_{j_1}\dots r_{j_n})^{\alpha} & \text{otherwise.} \end{cases}$$

We pay special attention to $Q_{\mathtt{j}}^{[\emptyset]}$; let $\mathcal{E}_n = \{\mathtt{j} \in [M]^n : Q_{\mathtt{j}}^{[\emptyset]} > 0\}$ (essentially containing those level n-nodes so that for each ancestor of the node its true that the assigned random variable was never 0, connecting cascades to Definiton 1.2). The random attractor corresponding to this construvtion is the compact set

$$\Lambda_{\mathcal{F}}(p) := \cap_{n=1} \cup_{\mathtt{i}\in\mathcal{E}_n} f_{\mathtt{i}}(B), \tag{A.6}$$

where B is as in 1.2.

Example A.2 (Homogeneous systems). We continue the previous example for homogeneous systems with M maps. In this case $\alpha = -\log_r(Mp)$, hence

$$Y_n^{[\mathtt{i}]} = \sum_{\mathtt{j}\in[M]^n} Q_{\mathtt{j}}^{[\mathtt{i}]} = \sum_{\mathtt{j}\in\mathcal{E}_n} (Mp)^{-n} = \frac{\#\mathcal{E}_n}{(Mp)^n} \tag{A.7}$$

This converges to the limit denoted by $Y^{[\mathtt{i}]}$, so that $\mathcal{E}(Y^{[\mathtt{i}]}) = 1$. Then the random multiplicative cascade measure $\mu^{[\mathtt{i}]}$ generated by the sequence of random variables $\{W^{[ij], j\in[M]^*}\}$ is as follows:

$$\mu^{[i]}([j]) = Q_j^{[i]} \cdot Y^{[ij]} = \begin{cases} 0, & \text{if } [ij] \cap \mathcal{E}_\infty = \emptyset \\ \frac{1}{(Mp)^k} \lim_{\ell\to\infty} \frac{\mathcal{E}_\ell^{[ij]}}{(Mp)^\ell}, & \text{otherwise} \end{cases}$$

Remark A.3 (Cascades and natural measure). In particular for the 2 dimensional Mandelbrot percolation, where we fix K and $M = K^2$, the natural projection of the cascade measure at the empty set (denoted by $\mu = \mu^{[\emptyset]}$) to the set can be compared to the natural measure appeared in [36] by Rams and Peres. Recall, from 4.3 that the natural measure is

$$\nu = \lim_{n\to\infty} \frac{\mathcal{L}eb|_{\Lambda_n}}{p^n \cdot W}, \tag{A.8}$$

where $W = \lim_{n\to\infty} \frac{\#\mathcal{E}_n}{(K^2\cdot p)^n}$, Let $\widehat{\mu} = \pi_*\mu$, then for any $Q_{\mathtt{j}}$ element of the natural K-adic partition (not entirely precisely) we have

$$\widehat{\mu}(Q_{\mathtt{j}}) = \mu([j]) = \frac{1}{(Mp)^k} \lim_{\ell\to\infty} \frac{\#\mathcal{E}_\ell^{[\mathtt{j}]}}{(Mp)^\ell},$$

whereas

$$\nu(Q_{\mathtt{j}}) = \frac{1}{Z(E)} \lim_{n\to\infty} \frac{\mathcal{L}eb|_{\Lambda_n}(Q_{\mathtt{j}})}{p^n} = \frac{1}{Z(E)} \lim_{\ell\to\infty} \frac{\#\mathcal{E}_\ell^{[\mathtt{j}]} M^{-\ell}}{p^\ell} = \frac{1}{Z(E)}\mu([\mathtt{j}]),$$

where recall that $\#\mathcal{E}_\ell^{[\mathtt{j}]}$ we denote the number of level $|\mathtt{j}| + \ell$ children of the node $\mathtt{j}$.

A.1. **Some results regarding coin tossing sets using the cascade measure.** The following results are again from [23]. First we state the result in a special case (see [23, Lemma 6.1]), with the same setup as in Example A.1, with the natural addition that $p > 1/M$ so the resulting set $\Lambda_{\mathcal{F}}(p) \neq \emptyset$ with positive probability, and the IFS satisfying the open set condition (OSC).

[1] In this case we have that $\mathbb{P}_*$ almost surely the push forward (from the symbolic space to the attractor) measure of μ: $\pi_*\mu$ is exact dimensional, with dimension:

$$\dim_H \pi_*\mu = \frac{\sum_{i=1}^m \mathbb{E}(W_i \log W_i)}{\sum_{i=1}^m \mathbb{E}(W_i)\log r_i} = \alpha = \dim_H \Lambda_{\mathcal{F}(p)}, \tag{A.9}$$

where recall, that α was chosen to be so that $\sum_{i=1}^m p r_i^\alpha = 1$ (which is also the a.s. dimension of the random attractor $\Lambda_{\mathcal{F}}(p)$ as written in the last inequality).

[2] Assume that the IFS satisfies OSC and has dense rotations (meaning that the group generated by the rotation matrices is dense in $SO(d,\mathbb{R})$), then assuming that we run the percolation process in Example A.1, we have

$$\dim_H \mathrm{proj}_V(\Lambda) = \min(k, \alpha), \tag{A.10}$$

where proj_V is the projection onto V, a k-dimensional subspace of $\mathbb{R}^d$.

Generally, when the system does not satisfy OSC (similarly to the deterministic case) we might have an additional term [23, Theorem 3.2 (i)], we only state the result of the Theorem partially: Let $\mathbb{P}$ be any distribution on $2^{[M]}$. Further, $\mathcal{P}$ be the partition of $[M]^{\mathbb{N}}$ according to the first coordinate, and B_π be the σ-algebra generated by $\pi^{-1}B(\mathbb{R}^d)$ (where $B(\mathbb{R}^d)$ is the Borel σ-algebra on $\mathbb{R}^d$). In this case we have that $\mathbb{P}_*$ (recall, that it was defined in A.4) almost surely $\pi_*\mu$ is exact dimensional, with dimension:

$$\frac{\mathbb{E}(\mathbf{H}_{\overline{\mu}}(\mathcal{P}|B_\pi)) + \sum_{i=1}^m \mathbb{E}(W_i \log W_i)}{\sum_{i=1}^m \mathbb{E}(W_i)\log r_i}, \tag{A.11}$$

where recall that $\overline{\mu}$ was defined in A.5 and $\mathbf{H}_{\overline{\mu}}(\mathcal{P}|B_\pi)$ is the conditional entropy of the partition $\mathcal{P}$ with respect to the σ-algebra B_π,

$$\mathbf{H}_{\overline{\mu}}(\mathcal{P}|B_\pi) = \int_{[M]^{\mathbf{N}}} -\sum_{A\in\mathcal{P}} \chi_A(i)\log(\mathbb{E}_{\overline{\mu}}(\chi_A|B_\pi))(i)d\overline{\mu}(i). \tag{A.12}$$

B. Substitution systems

In the literature sometimes substitution systems are used. It appears for example in [19], [13], [22] and [15]

This is a more general model than the above defined coin-tossing systems. In case of the substitution systems the following step is modified: In this case we have a probability distribution on the "possible formations". Namely, we consider a $H \subset \{0,1\}^{[M]}$ a set of subsets of $[M]$. For each element $h \in \{0,1\}^M$ of H we choose a probability $0 < p_h \leq 1$ forming a probability vector $\sum_{h \in H} p_h = 1$. In the first step we choose an element h of H according to the distribution. The random subset $\mathcal{E}_1 \subset [M]$ consists of those k for which $h_k = 1$. We define $\mathcal{E}_n$ inductively, repeating this step for the retained elements, analogously to the coin tossing system. We repeat the example of the Sierpiński carpet from [15].

Example B.1 (Random Sierpiński carpet, version 2)**.** Consider the matrices:

$$\mathbf{U}_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \mathbf{U}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and we let the random substitution be $\sigma(0) = \mathbf{U}_2$, and for $0 < p_1 \leq 1$ and $\mathbb{P}(\sigma(1) = \mathbf{U}_1) = p_1 = 1 - \mathbb{P}(\sigma(1) = \mathbf{U}_2)$. At time 0 we have a single 1 entry (so our 1×1 matrix looks like [1]), and at each step of the inductive construction we replace each 0s in the matrix by the matrix $\sigma(0) = \mathbf{U}_2$ and each 1 by the matrix $\sigma(1)$ independently for each entry. So that in the n-th step of the construction we have a $3^n \times 3^n$ matrix consisting of 0's and 1's.y

Clearly, this model is more general than the coin-tossing one, since the distribution $\mathbb{P}(h) = p^{\#\{k:h_k=1\}}(1-p)^{\#\{k:h_k=0\}}$ gives back the coin tossing system.

References

[1] I. Arhosalo, E. Järvenpää, M. Järvenpää, M. Rams, and P. Shmerkin. Visible parts of fractal percolation. *Proc. Edinb. Math. Soc., II. Ser.*, 55(2):311–331, 2012.

[2] K. B. Athreya and P. E. Ney. *Branching processes*, volume 196 of *Grundlehren Math. Wiss.* Springer, Cham, 1972.

[3] A. Banaji, A. Rutar, and S. Troscheit. Interpolating with generalized Assouad dimensions. Preprint, arXiv:2308.12975 [math.CA] (2023), 2023.

[4] B. Bárány, K. Simon, and B. Solomyak. *Self-Similar and Self-Affine Sets and Measures*. Math. Surv. Monogr. Providence, RI: American Mathematical Society (AMS), 2023.

[5] A. Berlinkov and E. Järvenpää. Porosities of Mandelbrot percolation. *J. Theor. Probab.*, 32(2):608–632, 2019.

[6] E. I. Broman, F. Camia, M. Joosten, and R. W. J. Meester. Dimension (In)equalities and Hölder continuous curves in fractal percolation. *J. Theor. Probab.*, 26(3):836–854, 2013.

[7] Z. Buczolich, E. Järvenpää, M. Järvenpää, T. Keleti, and T. Pöyhtäri. Fractal percolation is unrectifiable. *Adv. Math.*, 390:61, 2021. Id/No 107906.

[8] J. T. Chayes, L. Chayes, and R. Durrett. Connectivity properties of Mandelbrot's percolation process. *Probab. Theory Relat. Fields*, 77(3):307–324, 1988.

[9] L. Chayes. Aspects of the fractal percolation process. In *Fractal geometry and stochastics. Proceedings of a conference on fractal geometry and stochastics held at Finsterbergen, Germany, June 12-18, 1994*, pages 113–143. Basel: Birkhäuser, 1995.

[10] L. Chayes. On the length of the shortest crossing in the super-critical phase of Mandelbrot's percolation process. *Stochastic Processes Appl.*, 61(1):25–43, 1996.

[11] C. Chen, T. Ojala, E. Rossi, and V. Suomala. Fractal percolation, porosity, and dimension. *J. Theor. Probab.*, 30(4):1471–1498, 2017.

[12] F. M. Dekking. Random Cantor sets and their projections. In *Fractal geometry and stochastics IV. Proceedings of the 4th conference, Greifswald, Germany, September 8–12, 2008*, pages 269–284. Basel: Birkhäuser, 2009.

[13] F. M. Dekking and G. R. Grimmett. Superbranching processes and projections of random Cantor sets. *Probability Theory and Related Fields*, 78:335–355, 1988.

[14] F. M. Dekking and B. Kuijvenhoven. Differences of random Cantor sets and lower spectral radii. *J. Eur. Math. Soc. (JEMS)*, 13(3):733–760, 2011.

[15] F. M. Dekking and R. W. J. Meester. On the structure of Mandelbrot's percolation process and other random Cantor sets. *Journal of Statistical Physics*, 58(5-6):1109–1126, 1990.

[16] F. M. Dekking and K. Simon. On the size of the algebraic difference of two random Cantor sets. *Random Struct. Algorithms*, 32(2):205–222, 2008.

[17] H. Don. *Reflecting Walls and Dissipating Tiles: Billiards and Fractal Percolation*. PhD thesis.

[18] H. Don. New methods to bound the critical probability in fractal percolation. *Random Struct. Algorithms*, 47(4):710–730, 2015.

[19] K. J. Falconer. Random fractals. *Math. Proc. Camb. Philos. Soc.*, 100:559–582, 1986.

[20] K. J. Falconer. Projections of random cantor sets. *Journal of Theoretical Probability*, 2(1):65–70, 1989.

[21] K. J. Falconer and T. Feng. Fractal percolation on statistically self-affine carpets. Preprint, arXiv:2401.02829 [math.MG] (2024), 2024.

[22] K. J. Falconer and G. R. Grimmett. On the geometry of random Cantor sets and fractal percolation. *Journal of Theoretical Probability*, 5(3):465–485, 1992.

[23] K. J. Falconer and X. Jin. Exact dimensionality and projections of random self-similar measures and sets. *Journal of the London Mathematical Society. Second Series*, 90(2):388–412, jul 2014.

[24] J. M. Fraser. *Assouad dimension and fractal geometry*, volume 222 of *Camb. Tracts Math.* Cambridge: Cambridge University Press, 2021.

[25] J. M. Fraser, J. J. Miao, and S. Troscheit. The Assouad dimension of randomly generated fractals. *Ergodic Theory Dyn. Syst.*, 38(3):982–1011, 2018.

[26] J. M. Fraser and H. Yu. Assouad-type spectra for some fractal families. *Indiana Univ. Math. J.*, 67(5):2005–2043, 2018.

[27] Michael A. Klatt and Steffen Winter. Geometric functionals of fractal percolation. *Advances in Applied Probability*, 52(4):1085–1126, 2020.

[28] Michael A. Klatt and Steffen Winter. Almost sure convergence and second moments of geometric functionals of fractal percolation. *Advances in Applied Probability*, 56(3):927–959, 2024.

[29] R. D. Mauldin and S. C. Williams. Random recursive constructions: asymptotic geometric and topological properties. *Transactions of the American Mathematical Society*, 295(1):325–346, 1986.

[30] R. W. J. Meester. Connectivity in fractal percolation. *J. Theor. Probab.*, 5(4):775–789, 1992.

[31] P. Móra, K. Simon, and B. Solomyak. The Lebesgue measure of the algebraic difference of two random Cantor sets. *Indag. Math., New Ser.*, 20(1):131–149, 2009.

[32] V. Orgoványi and K. Simon. Projections of the random Menger sponge. *Asian J. Math.*, 27(6):893–936, 2023.

[33] V. Orgoványi and K. Simon. Interior points and Lebesgue measure of overlapping Mandelbrot percolation sets. Preprint, arXiv:2407.06750 [math.DS] (2024), 2024.

[34] M. E. Orzechowski. *Geometrical and Topological properties of Fractal Percolation*. Phd thesis, University of St Andrews.

[35] M. E. Orzechowski. A lower bound on the box-counting dimension of crossings in fractal percolation. *Stochastic Processes Appl.*, 74(1):53–65, 1998.

[36] Y. Peres and M. Rams. Projections of the natural measure for percolation fractals. *Israel Journal of Mathematics*, 214(2):539–552, 2016.

[37] M. Rams and K. Simon. The dimension of projections of fractal percolations. *J. Stat. Phys.*, 154(3):633–655, 2014.

[38] M. Rams and K. Simon. The dimension of projections of fractal percolations. *Journal of Statistical Physics*, 154(3):633–655, nov 2014.

[39] M. Rams and K. Simon. The geometry of fractal percolation. In *Geometry and analysis of fractals. Based on the international conference on advances of fractals and related topics, Hong Kong, China, December 10–14, 2012*, pages 303–323. Berlin: Springer, 2014.

[40] M. Rams and K. Simon. Projections of fractal percolations. *Ergodic Theory and Dynamical Systems*, 35(2):530–545, 2015.

[41] P. Shmerkin and V. Suomala. *Spatially independent martingales, intersections, and applications*, volume 251. American Mathematical Society, 2018.

[42] K. Simon and L. Vágó. Projections of Mandelbrot percolation in higher dimensions. In *Fractals, wavelets, and their applications. Contributions from the international conference and workshop on fractals and wavelets, Kerala, India, November 9–12, 2013*, pages 175–190. Cham: Springer, 2014.

[43] K. Simon and L. Vágó. Fractal percolations. In *Dynamical systems. Simons Semester in Banach Center, 2015. Papers based on the courses delivered at the Semester, Będlewo, Poland, September and November, 2015*, pages 183–196. Warsaw: Polish Academy of Sciences, Institute of Mathematics, 2018.

[44] S. Troscheit. On the dimensions of attractors of random self-similar graph directed iterated function systems. *J. Fractal Geom.*, 4(3):257–303, 2017.

[45] S. Troscheit. The quasi-Assouad dimension of stochastically self-similar sets. *Proc. R. Soc. Edinb., Sect. A, Math.*, 150(1):261–275, 2020.

[46] P. van der Wall. *Random substitutions and fractal percolation.* Phd thesis, Delft University of Technology, 2003. Available at https://repository.tudelft.nl/record/uuid:1d64cec8-d06b-4cfb-8ade-e441f63039b4.

[47] D. G. White. On the value of the critical point in fractal percolation. *Random Struct. Algorithms*, 18(4):332–345, 2001.