Orthogonal projections of the random Menger sponge

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joint work with Károly Simon

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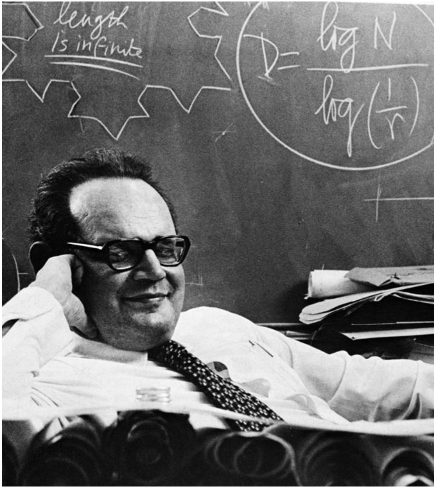
Benoit Mandelbrot



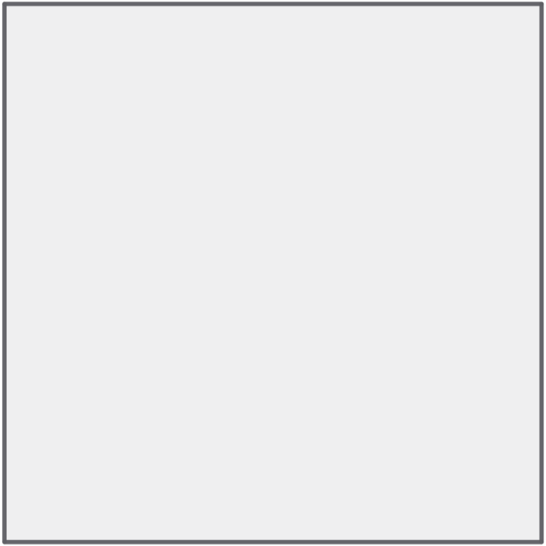
Construction of the (homogeneous) Mandelbrot percolation fractal $\Lambda_d(M, p)$

$\mathbb{R}^d$

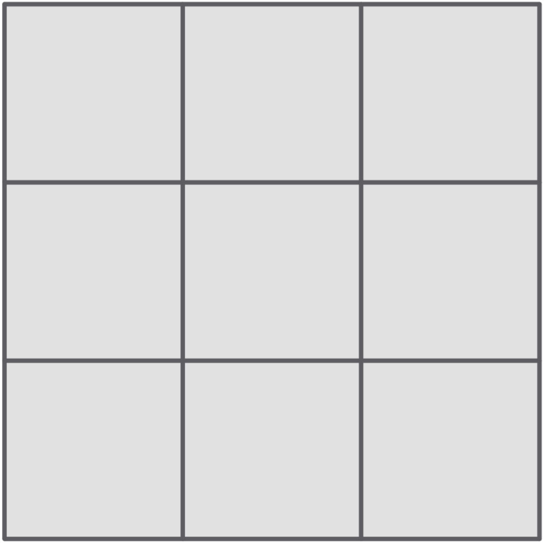
- $M \in \mathbb{N} \setminus \{0, 1\}$: division parameter
- $p \in [0, 1]$: probability

$\mathbb{P}($  $) = p$ and $\mathbb{P}($  $) = 1 - p$.

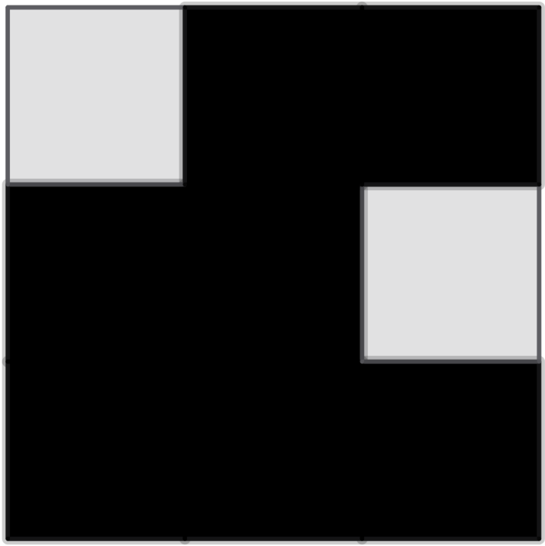
Construction

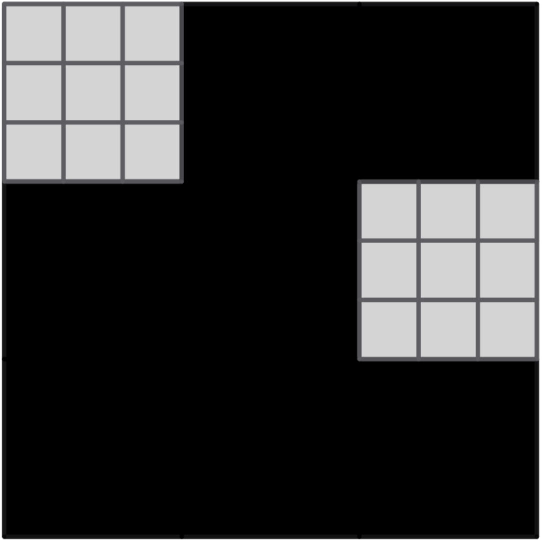


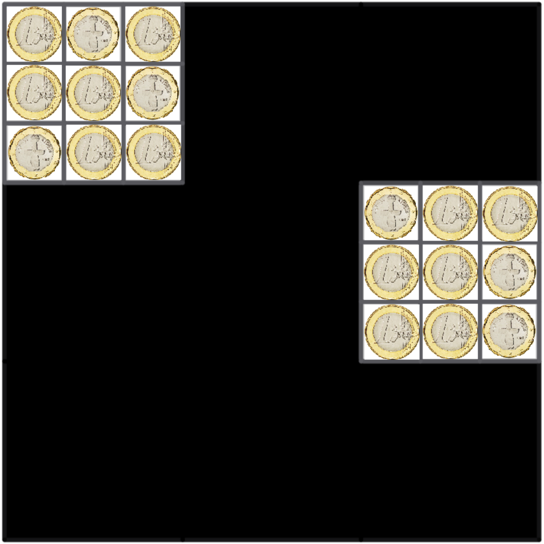
Construction

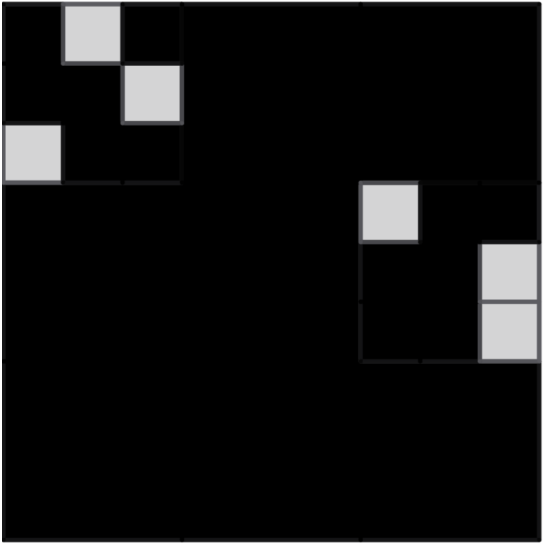


Construction





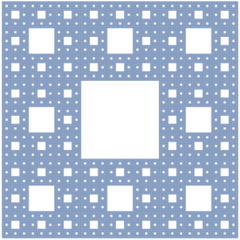




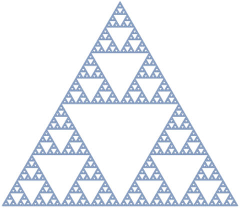
Hausdorff dimension of a set (examples)

$\Lambda_d(M,p)$: Mandelbrot percolation in $\mathbb{R}^d$ with parameters M and p.
Falconer, and Mauldin and Williams:

$$\dim\Lambda_d(M,p) = \frac{\log \mathbb{E}(\#\text{retained level-1 squares})}{\log M}, \dim\Lambda_2(3,p) = \frac{\log 9 \cdot p}{\log 3}$$



$\frac{\log 8}{\log 3}$



$\frac{\log 3}{\log 2}$

Size of $\Lambda_d(M, p)$

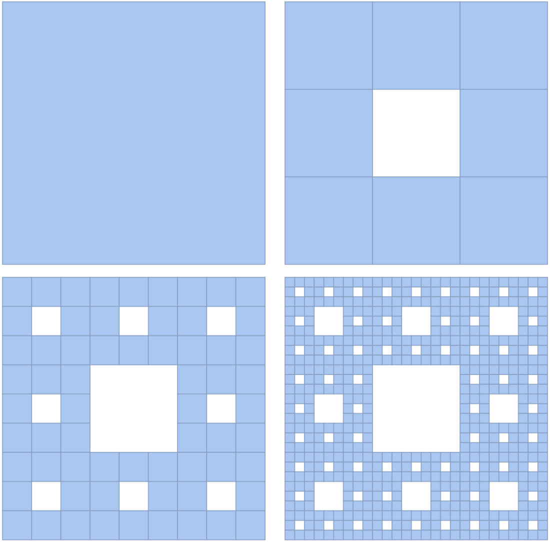
1. Chayes-Chayes-Durrett: There exits a p_{crit}
   - $p < p_{\text{crit}}$: $\Lambda_2(3, p)$ is totally disconnected;
   - $p \geq p_{\text{crit}}$: the opposite sides are connected with positive probability.

   $M = 3$: If $p > \frac{1}{3}$ then $dim_H(\Lambda_2(3, p)) > 1$ and $(0.784 < p_{\text{crit}} < 0.94)$.

2. Simon-Rams (2-dim), Simon-Vágó (d-dim)
   $\dim_H(\Lambda_d(M, p)) > 1$ $\left(\longleftrightarrow p > M^{d-1}\right) \longrightarrow$ for almost all realizations it holds that simoultaneously to all lines of $\mathbb{R}^d$ the orthogonal projection contains an interval.

Spec. if $p > \frac{1}{3}$, then $\Lambda_2(3, p)$ contains an interval almost surely.

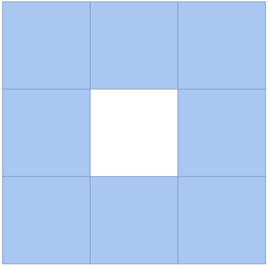
Sierpiński carpet



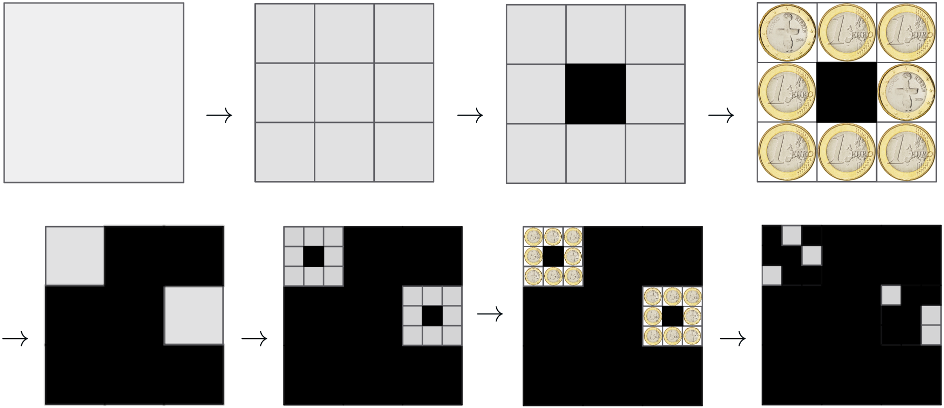
Inhomogeneous Mandelbrot percolation, random Sierpiński carpet

$0 < p < 1$

$\mathbb{P}($$) = p$ and $\mathbb{P}($$) = 1 - p$.



Inhomogeneous Mandelbrot percolation, random Sierpiński carpet



Random Sierpiński carpet

$\dim_H(\Lambda_d^{\mathfrak{D}}(M,p)) = \frac{\log(\mathbb{E}(\#\text{retained level 1 cubes}))}{-\log(\text{contraction ratio})}$ a.s. conditioned on non-extinction.

Spec. for the random Sierpiński carpet ($\mathcal{S}_p$) $\dim_H(\mathcal{S}_p) = \frac{\log 8 \cdot p}{\log 3}$.

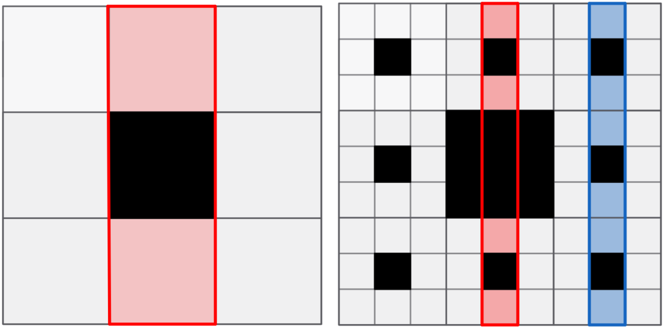
1. Falconer-Grimmett (1992): projections to the coordinate axes.
2. Simon and Vágó (2-dim): For any fixed rational direction α, there exist a $p_\alpha > \frac{3}{8}$ such that for $p \in (\frac{3}{8}, p_\alpha)$ the orthogonal projection of $\mathcal{S}_p$ to direction α has empty interior almost surely.

Heuristic explanation of the gap

2 factors:

- "Regular overlapping structure" in the projected system.
- "Inhomogenity in space" of the original system.

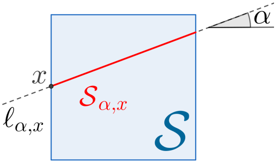
Example—projection to the coordinate axes:



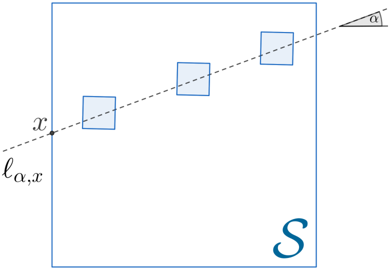
$p = \frac{3}{8} + \varepsilon$
m-times middle columns: $p^{k+m} 2^m 3^k \to 0$ as $m \to \infty$.

Other rational directions

$\mathcal{S}$: DETERMINISTIC Sierpiński carpet;
$\ell_{\alpha,x}$: line through x with angle α;
$\mathcal{S}_{\alpha,x}$: the slice of $\mathcal{S}$ through x with angle α;
$\mathcal{N}_{\alpha,x,n}$: number of level-n squares in $\mathcal{S}$ intersecting $\ell_{\alpha,x}$.

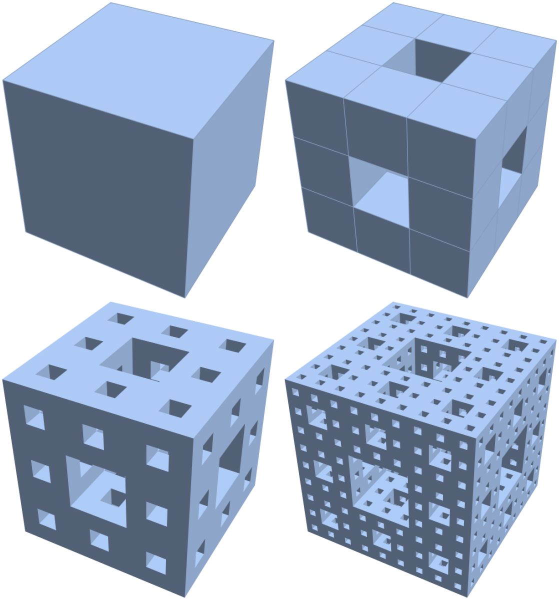


Marstrand: for $\mathcal{L}$eb almost every (α, x) pair, $\dim_B \mathcal{S}_{\alpha,x} = \frac{\log 8}{\log 3} - 1$.
Simon-Manning (2013): For any fixed α, if $\tan\alpha$ is rational then for $\mathcal{L}$eb almost every x: $\dim_B \mathcal{S}_{\alpha,x} = c_\alpha < \frac{\log 8}{\log 3} - 1$.



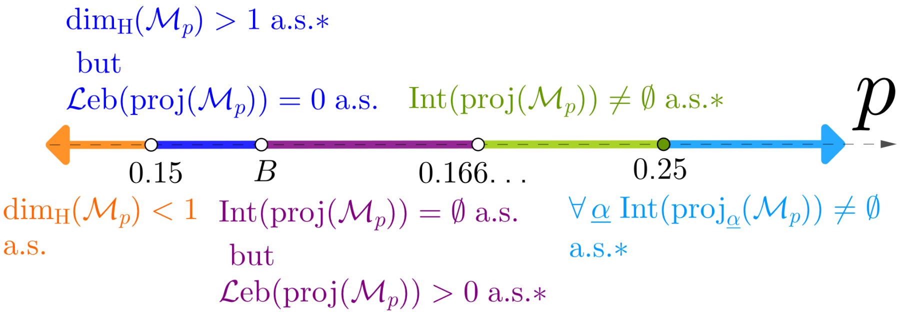
Choose ε such that $c_\alpha + \varepsilon < \frac{\log 8}{\log 3} - 1$ then:
$\mathcal{N}_{\alpha,x,n} < 3^{n(c_\alpha+\varepsilon)} < (\frac{8}{3} - \delta)^n$.

Menger sponge



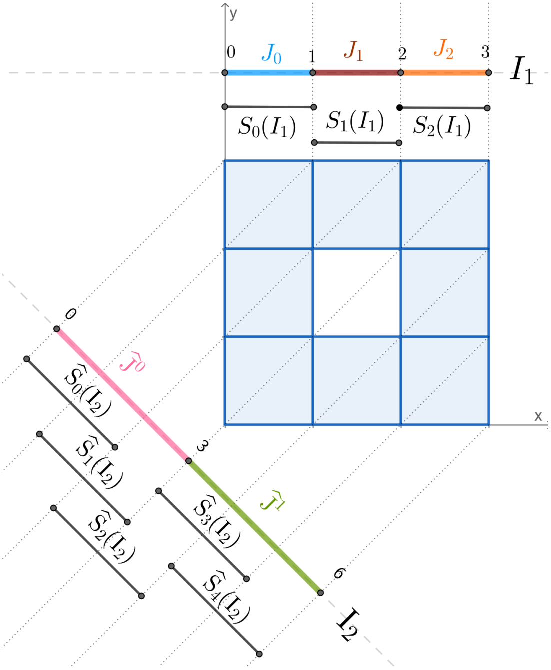
Orthogonal projection of the random Menger sponge

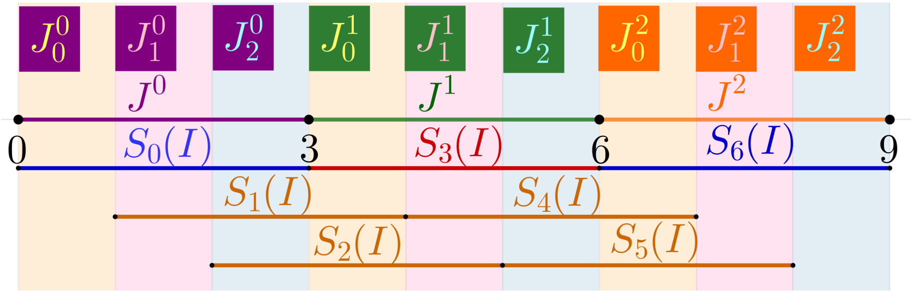
$\mathcal{M}_p$: random Menger sponge with parameter p;
proj: projection to the space diagonal of the unit cube;
$\text{proj}_{\underline{\alpha}}$: projection of the form $\underline{x} \to \underline{\alpha}\,\underline{x}$.



* - conditioned on non-extinction.
$0.15 < B < 0.1514\ldots$

Thank you for your attention!


y
0
J_0
1
J_1
2
J_2
3
I_1
$S_0(I_1)$
$S_1(I_1)$
$S_2(I_1)$
x
0
$\widehat{J}^0$
3
$\widehat{J}^1$
6
I_2
$\widehat{S}_0(I_2)$
$\widehat{S}_1(I_2)$
$\widehat{S}_2(I_2)$
$\widehat{S}_3(I_2)$
$\widehat{S}_4(I_2)$



J_0^0
J_1^0
J_2^0
J_0^1
J_1^1
J_2^1
J_0^2
J_1^2
J_2^2
J^0
J^1
J^2
0
3
6
9
$S_0(I)$
$S_3(I)$
$S_6(I)$
$S_1(I)$
$S_4(I)$
$S_2(I)$
$S_5(I)$