Orthogonal projections of the random Menger sponge

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joint work with Károly Simon

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Motivation

Are there any deterministic, self-similar set on the line with positive Lebesgue measure but empty interior?

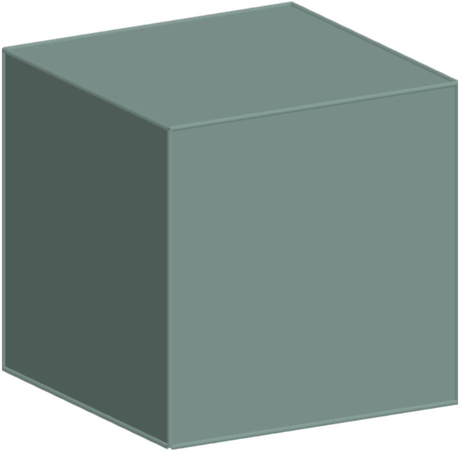
There is a random ~~deterministic~~, self-similar set on the line with positive Lebesgue measure but empty interior.

Choose $0 \leq p \leq 1$.

Take a biased coin such that

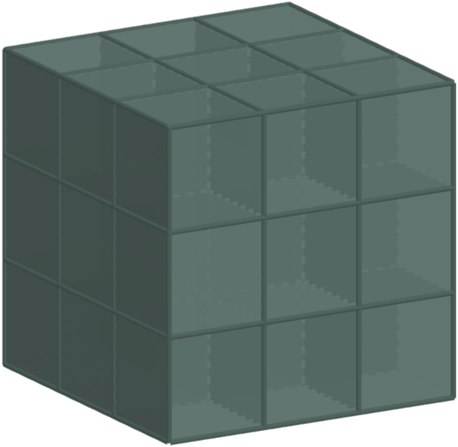
$\mathbb{P}($  $) = p$ and $\mathbb{P}($  $) = 1 - p$.

Construction of the random Menger sponge



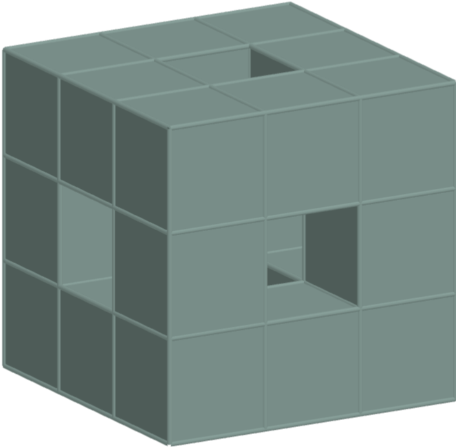
1. Unit cube.

Construction of the random Menger sponge



1. Unit cube.
2. Division into 3^3 congruent cubes.

Construction of the random Menger sponge



1. Unit cube.
2. Division into 3^3 congruent cubes.
3. Deletion.

Choose $0 \leq p \leq 1$.

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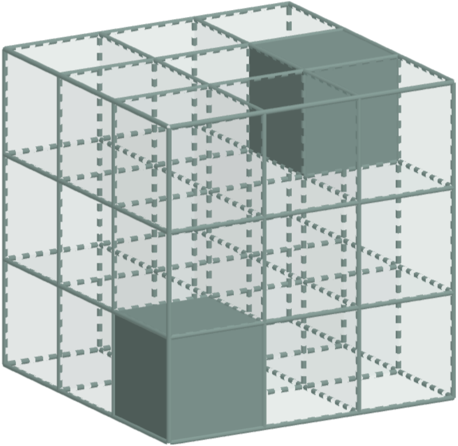
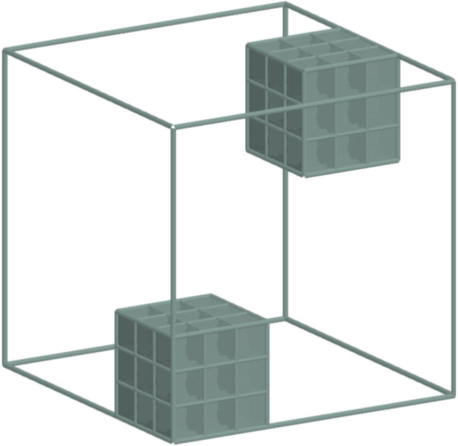
Construction of the random Menger sponge



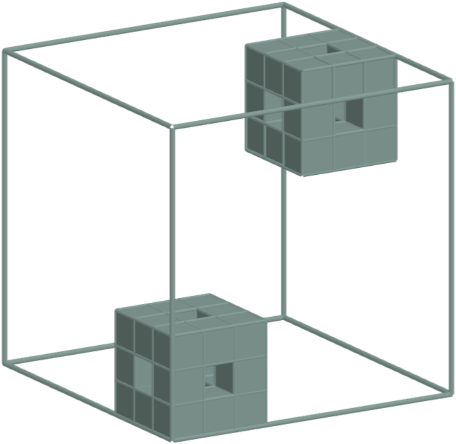
1. Unit cube.
2. Division into 3^3 congruent cubes.
3. Deletion.
4. Toss the coin for each independently.
   Heads $\rightarrow$ retain.
   Tails $\rightarrow$ discard.

Construction of the random Menger sponge



1. Unit cube.
2. Division into 3^3 congruent cubes.
3. Deletion.
4. Toss the coin for each independently.
   Heads $\to$ retain.
   Tails $\to$ discard.
5. Repetition ad infinitum or until we do not have any retained cubes left.

Construction of the random Menger sponge



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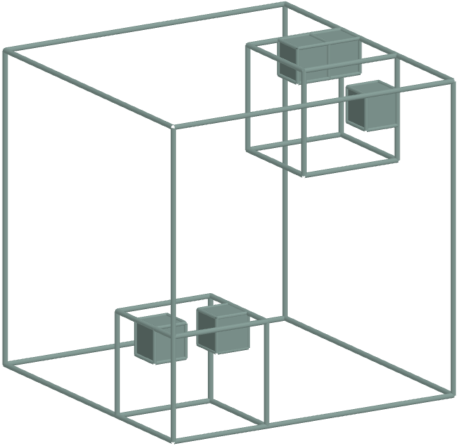

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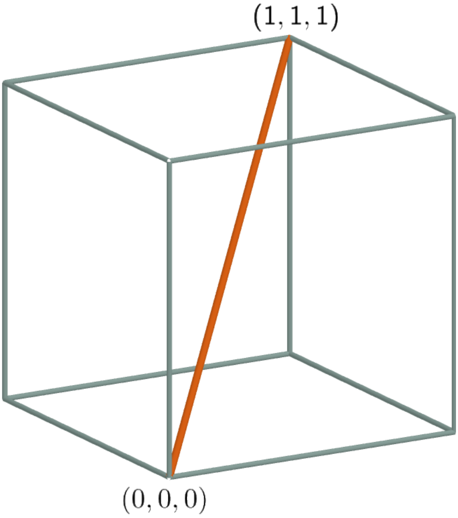
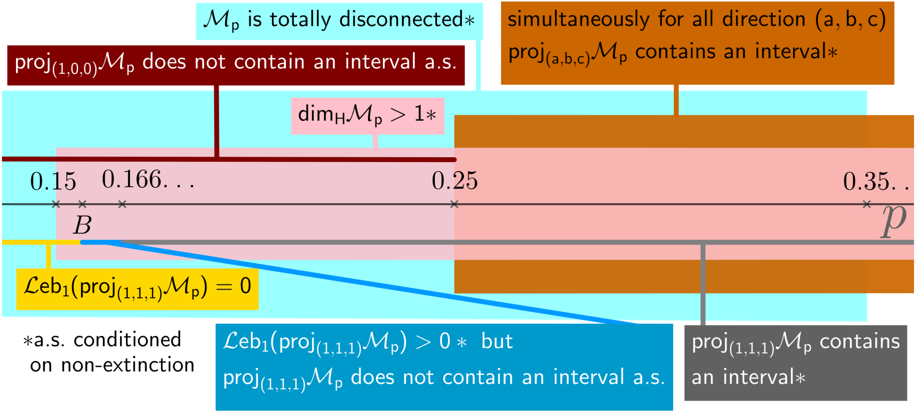
Projection of the random Menger sponge



- Choose p from the interval $(0.1514\ldots, 0.16\ldots)$
- project it to the space diagonal of the unit cube.

$\rightarrow$ random, self-similar set on the line, with positive Lebesgue measure almost surely conditioned on non-extinction and empty interior almost surely.

Parameter intervals



$$0.15 < B \leq 0.1514\ldots$$

Orthogonal projections of the random Menger sponge

Vilma Orgoványi[2] and Károly Simon[1,2,3]

Abstract

Using a similar random process to the one which yields the fractal percolation sets, starting from the deterministic Menger sponge we get the random Menger sponge. We examine its orthogonal projections from the point of Hausdorff dimension, Lebesgue measure and existence of interior point.

Fractal percolation

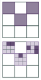


Figure 1.

Figure 2.

The (homogeneous) fractal percolation set $F^{(d)}_{(M,p)}$ is a two-parameter (M,p) family of random fractals in $\mathbb{R}^d$. I present the definition for $M=3$ and $d=2$.

I First we take the unit square and divide it to 3^2 congruent subsquares.

II For each of this square we toss a biased coin independently, that results in head with probability p and in tails with probability $1-p$.

III We retain a cube if the coin tossing results in head and discard otherwise.

IV To obtain $F^{(2)}_{(3,p)}$ we repeat this process in every retained square independently ad infinitum or until we do not have any retained squares left. The later case is called extinction.

A level 1 and 2 approximations of a possible realization are illustrated in Figure 1, the retained squares are colored purple.

- Analogously, in $\mathbb{R}^d$ the same process results in a d-dimensional fractal percolation set, denoted by $F^{(d)}_{(M,p)}$.
- For the inhomogeneous fractal percolation we choose a subset of level-1 squares which we consider to be always discarded and run the process only on the remaining squares. For example in the case of the random Sierpiński carpet the middle square is always discarded and can be regarded as a square with retention probability 0 as it is indicated in Figure 2.

Construction of the random Menger sponge

The random Menger sponge with parameter p – denoted by $\mathcal{M}_p$ – is a special inhomogeneous fractal percolation. The cubes that are always discarded are those which are not contained in the first approximation of the (deterministic) Menger sponge (see the figure).

Motivation

Rams and Simon's theorem [3] states that $\dim_H\left(F^{(2)}_{(M,p)}\right) > 1$ implies the existence of intervals simultaneously in all orthogonal projections to all lines, almost surely conditioned on non-extinction. However, this assertion does not always hold for two dimensional inhomogeneous fractal percolation sets as shown in [6]. The higher dimensional version of Rams and Simon's theorem was proved in [5]. Among other things, here we prove that $\dim_H(\mathcal{M}_p) > 1$ does not even imply that the Lebesgue measure of all orthogonal projections are positive.

Results

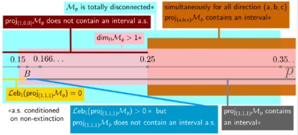
Let $\text{proj}_{(a,b,c)}(x,y,z) := ax + by + cz$. We consider the special projections

- $\text{proj}_{(1,0,0)}$, which is the orthogonal projection to the x-coordinate axis, and
- $\text{proj}_{(1,1,1)}$, a rescaled version of the orthogonal projection to the space diagonal of the unit cube $[0,1]^3$.

First, generally about the random Menger sponge $\mathcal{M}_p$ we can say the following:

a) Let $p \in (0.25, 1]$. Then simultaneously for all directions (a,b,c) the projection contains an interval almost surely conditioned on non-extinction.

This is interesting when $p < 0.35\ldots$ since in this case the Menger sponge itself is totally disconnected. Also the bound 0.25 is sharp because for $p < 0.25$ the $\text{proj}_{(1,0,0)}$-projection does not contain an interval almost surely.



The parameter intervals for p in the case of the random Menger sponge.

About $\text{proj}_{(1,1,1)}(\mathcal{M}_p)$ we proved the following. There exists $0.15 < B \leq 0.166\ldots$ such that:

b) For $p \in (0.15, B)$ the Lebesgue-measure of $\text{proj}_{(1,1,1)}(\mathcal{M}_p)$ is zero almost surely, despite the fact that $\mathcal{M}_p$ has Hausdorff dimension greater than 1 almost surely conditioned on non-extinction (see [1]).

c) For $p \in (B, 0.1666\ldots)$, conditioned on non-extinction, $\text{proj}_{(1,1,1)}(\mathcal{M}_p)$ is a set of positive Lebesgue measure which contains no interior points almost surely.

d) For $p \in (0.166\ldots, 1]$, conditioned on non-extinction, $\text{proj}_{(1,1,1)}(\mathcal{M}_p)$ contains an interval almost surely.

Conclusions

The existence of the c) (dark blue) phase, when the projection has positive Lebesgue measure (almost surely conditioned on non-extinction) but it does not contain an interval almost surely is especially interesting. Not only because we haven't seen such behaviour of a fractal percolation set before, but because it is an open question whether there exists a deterministic one-dimensional, self-similar set which does not contain an interval although its Lebesgue measure is positive. The question about the deterministic case is still unanswered. However, apparently there exists such a random self-similar set, namely the projection of the random Menger sponge $\mathcal{M}_p$ if we choose the p parameter right.

Further information

Most of the results presented here are special cases of our more general results stated for random percolation self-similar Cantor sets introduced by Falconer and Jin in [2, Section 6]. These more general results can be found in our paper *Projections of the random Menger sponge* [4].

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Thank you for your attention!