Orthogonal projections of the random Menger sponge

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joint work with Károly Simon

June 13, 2022

Hausdorff dimension of a set (definition)

Definition

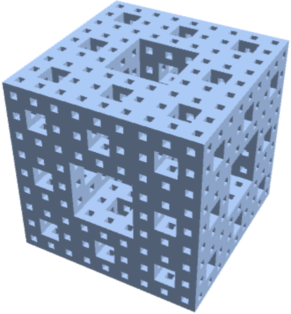
$$\mathcal{H}^t(E) := \lim_{\delta \to 0} \{ \underbrace{\inf \left\{ \sum_{i=1}^{\infty} |A_i|^t : E \subset \bigcup_{i=1}^{\infty} A_i, |A_i| \leq \delta \right\}}_{\mathcal{H}^t_\delta(E)} \}$$

$|A| :=$ diameter(A)

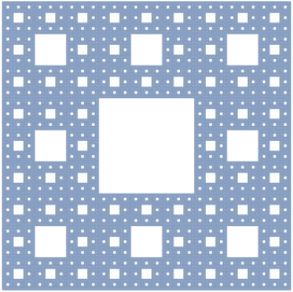
Definition

$$\dim_H(E) := \inf\{t : \mathcal{H}^t(E) > 0\} = \sup\{t : \mathcal{H}^t(E) < \infty\}$$

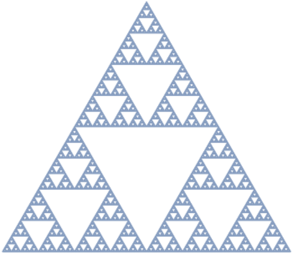
Hausdorff dimension of a set (examples)



$\frac{\log 20}{\log 3}$

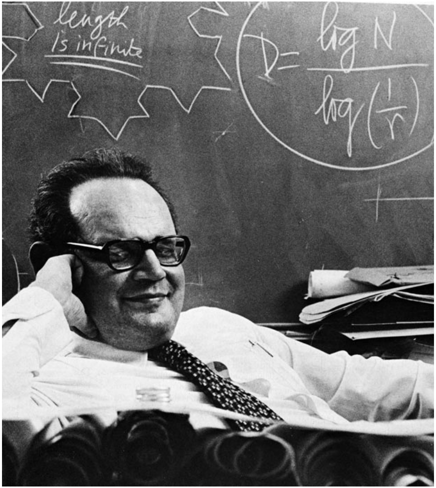


$\frac{\log 8}{\log 3}$



$\frac{\log 3}{\log 2}$

Benoit Mandelbrot



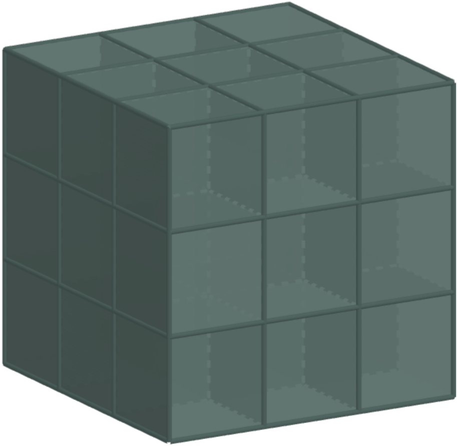
éskdf

Construction of the (homogeneous) Mandelbrot percolation fractal $\Lambda_d(M, p)$

- $d \in \mathbb{N} \setminus \{0\}$: dimension
- $M \in \mathbb{N} \setminus \{0, 1\}$: I don't know yet
- $p \in [0, 1]$: probability

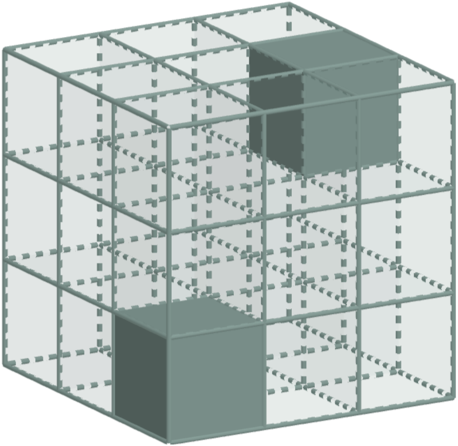
$\mathbb{P}($  $) = p$ and $\mathbb{P}($  $) = 1 - p$.

Construction of $\Lambda_3(3, p)$



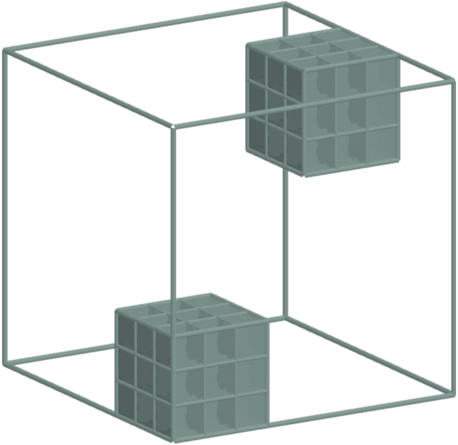
- I Unit cube.
- II Division into 3^3 congruent cubes.

Construction of $\Lambda_3(3, p)$



- I Unit cube.
- II Division into 3^3 congruent cubes.
- III Toss the coin for each independently.
  Heads $\rightarrow$ retain.
  Tails $\rightarrow$ discard.

Construction of $\Lambda_3(3, p)$



- I Unit cube.
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  Heads $\rightarrow$ retain.
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- IV Repetition ad infinitum or until we do not have any retained cubes left.

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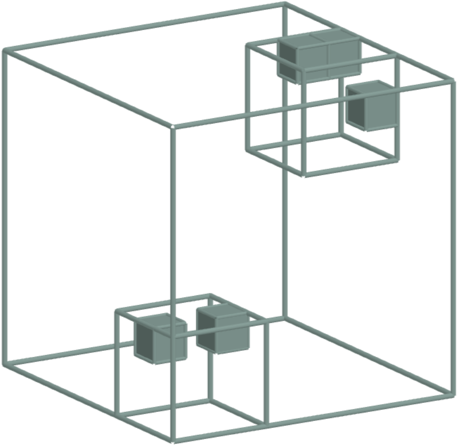

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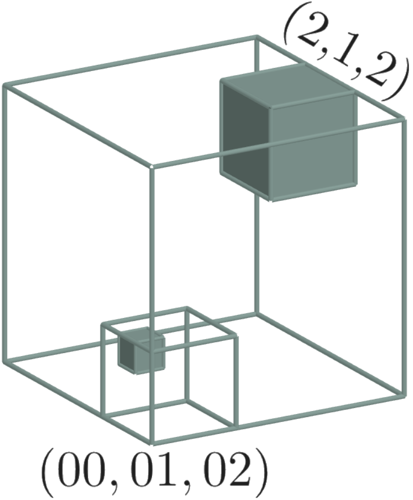
Some notations



$\mathcal{Q}_n := \{0, \ldots, M-1\}^{3n}$

$\mathcal{I} := [0,1]^3$ and for $(\underline{i}, \underline{j}, \underline{k}) \in \mathcal{Q}_n$:

$$\mathcal{I}_{(\underline{i},\underline{j},\underline{k})} := \left[\sum_{\ell=1}^{n} i_\ell 3^{-\ell}, \sum_{\ell=1}^{n} i_\ell 3^{-\ell} + 3^{-n}\right] \times \left[\sum_{\ell=1}^{n} j_\ell 3^{-\ell}, \sum_{\ell=1}^{n} j_\ell 3^{-\ell} + 3^{-n}\right] \times \left[\sum_{\ell=1}^{n} k_\ell 3^{-\ell}, \sum_{\ell=1}^{n} k_\ell 3^{-\ell} + 3^{-n}\right]$$

$\mathcal{E}_n := \{(\underline{i}, \underline{j}, \underline{k}) \in \mathcal{Q}_n :$ the cube of index $(\underline{i}, \underline{j}, \underline{k})$ is retained at level $n\}$

Size of $\Lambda_d(M, p)$

Q1 $\dim_{\mathrm{H}}(\Lambda_d(M, p)) = ?$

Falconer: $\dim_H(\Lambda_d(M, p)) = \frac{\log \mathbb{E}(\#\mathcal{E}_1)}{\log M} = \frac{\log M^d p}{\log M}$ a.s. conditioned on non-extinction.

Q2 Size of the orthogonal projections to lines?

Simon and Rams (2-dim), Simon-Vágó (d-dim):

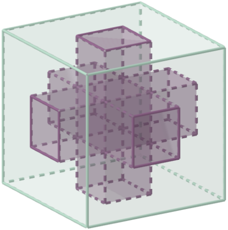
$\dim_H(\Lambda_d(M, p)) > 1$ $\left(\longleftrightarrow p > M^{d-1}\right) \longrightarrow$

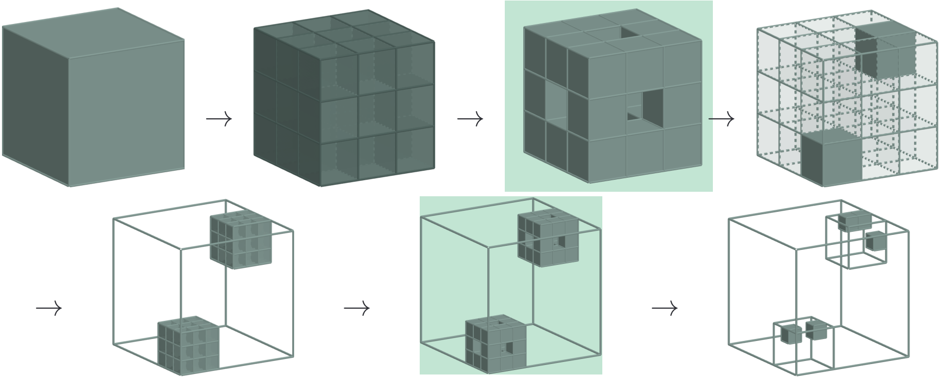
for almost all realizations, simoultaneously to all lines of $\mathbb{R}^d$ the orthogonal projection contains an interval.

Example: Random Menger sponge

Construction of the random Menger sponge $\mathcal{M}_p$

$$\mathfrak{D} := \{ (1,1,2), (1,0,1),(0,1,1),(1,1,1),(2,1,1),(1,2,1), (1,1,0)\}$$





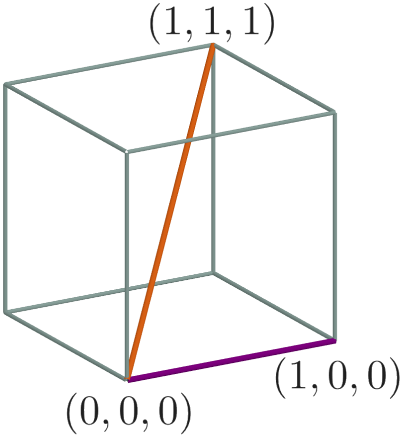
Inhomogeneous Mandlebrot percolation continued

Q1 $\dim_H(\Lambda_d^{\mathfrak{D}}(M,p)) = \frac{\log(\mathbb{E}(\#\text{retained level 1 cubes}))}{-\log(\text{contraction ratio})} = \frac{\log(M^d - \#\mathfrak{D})p}{\log M}$ a.s. conditioned on non-extinction.

Q2 Simon and Rams and Simon and Vágó (2-dim): It is possible that the dimension is greater than 1 a.s. condirtioned on non extinction BUT the projection to some direction does not contains an interval a.s. (different to the homogeneous case)

Projections

$\mathrm{proj}_{(a,b,c)} : \mathbb{R}^3 \to \mathbb{R}$, $\mathrm{proj}_{(a,b,c)}(x, y, z) := ax + by + cz$
$\mathrm{proj}_{(1,1,1)}$ and $\mathrm{proj}_{(1,0,0)}$



Orthogonal projections of the random Menger sponge

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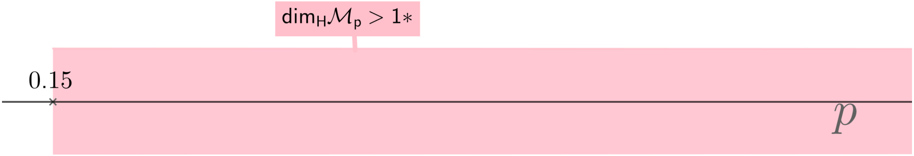

$*$a.s. conditioned
on non-extinction

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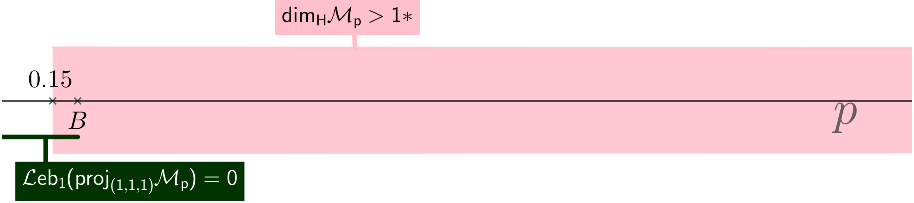

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$0.15 < B \leq 0.1514\ldots$

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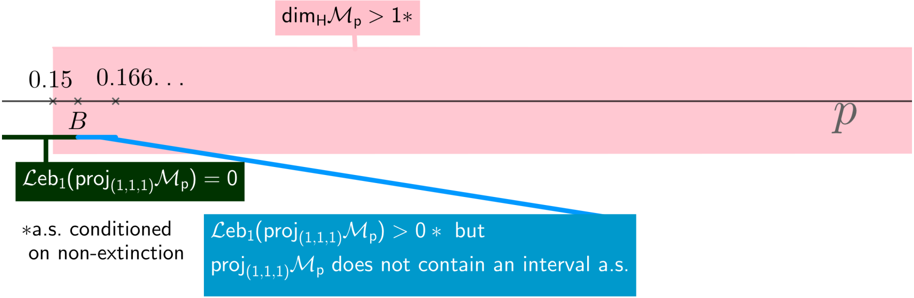

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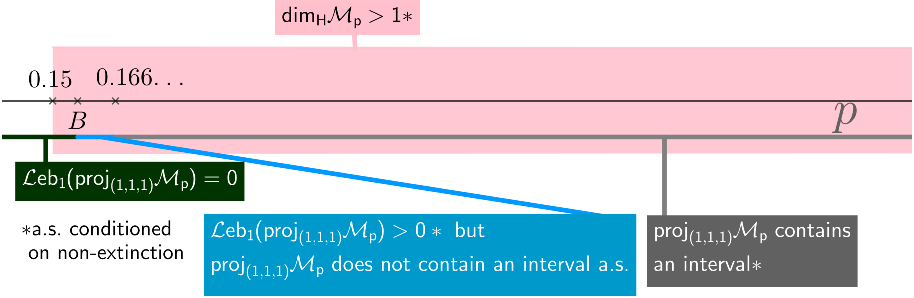

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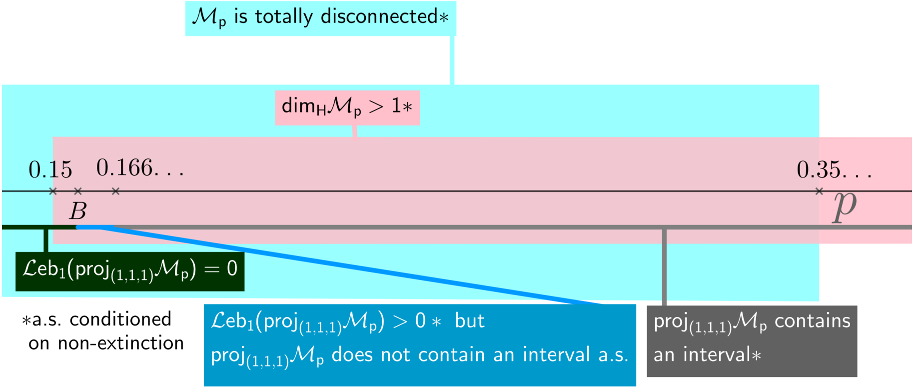

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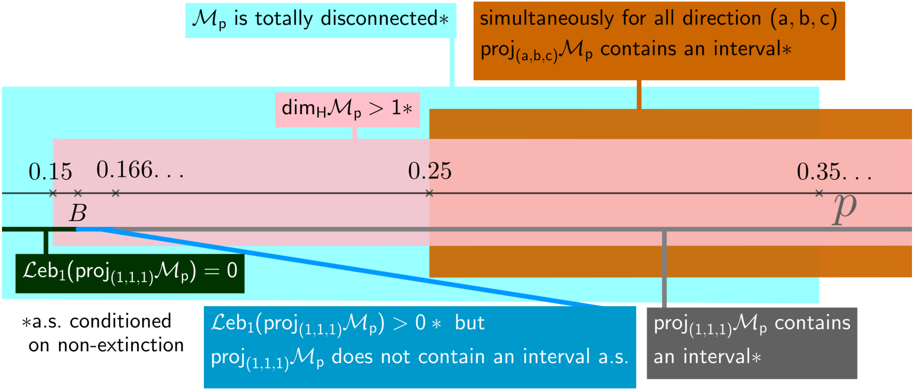

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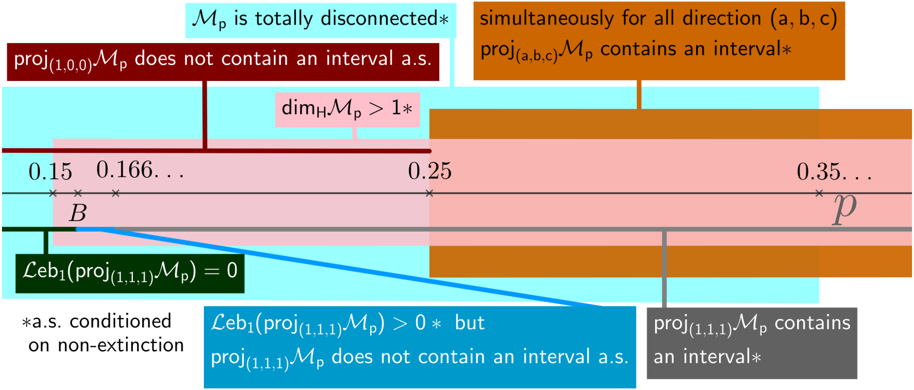


$0.15 < B \leq 0.1514\ldots$

Acknowledgement

The ideas of the proofs are coming from the following papers:

- Falconer and Grimmett: On the Geometry of Random Cantor Sets and Fractal Percolation.
- Simon and Dekking: On the Size of the Algebraic Difference. of Two Random Cantor Sets
- ...

Idea of the proof

STATEMENT: If $p > \frac{1}{6} = 0.166\ldots$, then $\text{proj}_{(1,1,1)}(\mathcal{M}_p)$ contains an interval almost surely conditioned on $\mathcal{M}_p$ being not empty.

part 1 $\mathbb{P}(\text{Int}(\text{proj}_{(1,1,1)}(\mathcal{M}_p)) \neq \emptyset) > 0$ implies $\mathbb{P}(\text{Int}(\text{proj}_{(1,1,1)}(\mathcal{M}_p)) \neq \emptyset | \mathcal{M}_p \neq \emptyset) = 1$.

part 2 Show $\mathbb{P}(\text{Int}(\text{proj}_{(1,1,1)}(\mathcal{M}_p)) \neq \emptyset) > 0$.

Part 1

$$\mathbb{P}(\mathrm{Int}(\mathrm{proj}_{(1,1,1)}(\mathcal{M}_p)) \neq \emptyset) > 0$$

implies

$$\mathbb{P}(\mathrm{Int}(\mathrm{proj}_{(1,1,1)}(\mathcal{M}_p)) \neq \emptyset | \mathcal{M}_p \neq \emptyset) = 1.$$

Ingredients:

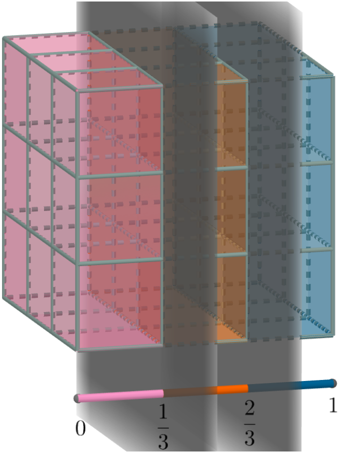
A Statistical self-similarity.

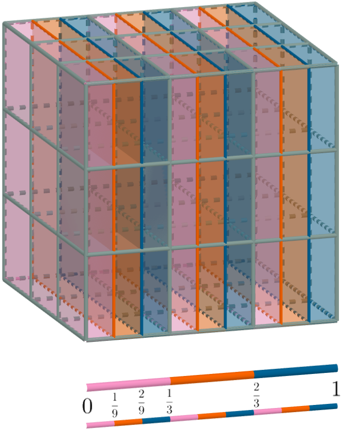
B With probability 1 conditioned on non extinction $\#\mathcal{E}_n$ (the number of retained level n squares) tends to infinity.

C The projection does not contain an interval if and only if for every n its intersection with every level n retained squares it does not contain an interval.

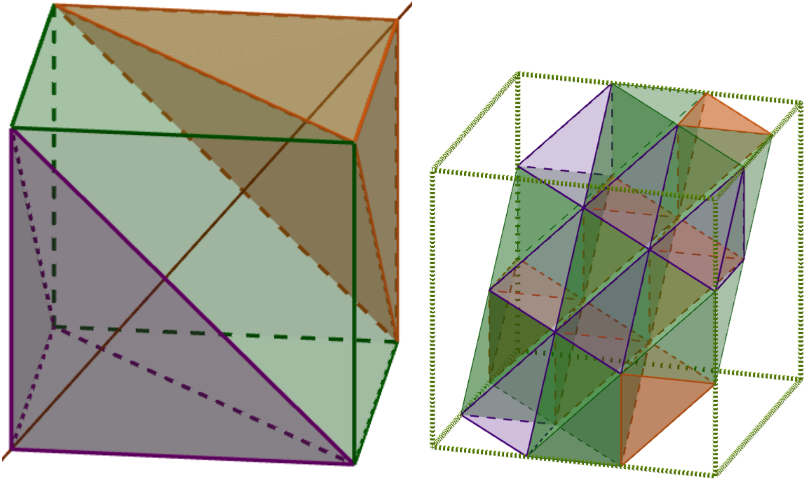
Part 2, Main idea, projection to the coordinate axes

Let $p > \frac{1}{4}$.





Part 2, Main idea, generalization



Thank you for your attention!